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Regions of Stability, Equivalence Theorems and the Courant-Friedrichs-Lewy Condition

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Summary. The celebrated CFL condition for discretizations of hyperbolic PDEs is shown to be equivalent to some results of Jeltsch and Nevanlinna concerning regions of stability of k-step, m-stage linear methods for the integration of ODEs. We characterize the methods for the numerical integration of the model equation $u_t = u_x$ which are weakly stable when the mesh-ratio takes the maximum value allowed by the CFL condition. We provide new equivalence theorems between stability and convergence, which improve on the classical results.

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1. Introduction

When discretizing hyperbolic problems by means of explicit algorithms, the famous Courant-Friedrichs-Lewy (CFL) condition [2] concerning the domains of dependence imposes upper bounds on the mesh-ratio $\Delta t/\Delta x$ which must necessarily be satisfied if the discretizations are to be convergent. The essentially *geometric* way in which those bounds are derived is in sharp contrast with the *algebraic* flavour usually found in other stability theories in Numerical Analysis. In this paper it is shown that, in spite of such a contrast, the CFL condition is in fact deeply related to some known "algebraic" stability conditions. More precisely, this relation is as follows. Assume first that we take for granted the CFL principle, i.e. that for convergent methods the numerical domain of dependence contains the theoretical domain of dependence. Then this principle can be used to provide a new proof of the following results of Jeltsch and Nevanlinna [4, 5]:

(A) If the disk $\{\mu \in \mathbb{C} : |\mu + r| \leqq r\}$ is contained in the region of absolute stability of a consistent, *explicit* linear k-step, m-stage method for the numerical integration of ODEs, then $r \leqq m$.

(B) If the interval $\{iy: y\in[-r,r]\}$ is contained in the closure of the region of absolute stability of a consistent, *explicit* linear k-step, m-stage method for the numerical integration of ODEs, then $r\leqq m$.

Conversely, and at least in the case of the model equation $u_t=u_x$, (A) and (B) can be applied to derive algebraically the upper bounds on $\Delta t/\Delta x$ provided by the CFL principle.

The contents of the paper is as follows. Section 2 is devoted to the presentation of some results on *weak stability* which are required later in the paper. The CFL principle is recalled in Sect. 3. Section 4 contains the new proofs of the Theorems (A) and (B) above. In Sect. 5 we show how (A) and (B) lead to the CFL bounds. We also present some new proofs of the fact that convergent schemes are stable. These proofs work with very mild definitions of convergence and thus improve on the classical Lax-Richtmyer equivalence theorem. The final Section provides a characterization of the schemes for which the CFL condition is *sufficient* for weak stability.

It should be mentioned that Theorem 4.1 of [13] can also be interpreted as a relationship between a CFL condition and the disk result (A).

2. Weak Stability

We consider constant coefficient, pure initial value problems in $-\infty<x<\infty$, $0<t\leqq T^*<\infty$. We denote by τ, h the step-sizes in time and space respectively and assume that as τ, h vary the ratio $r=\tau h^{-M}$ remains constant (M a positive integer). The translation operator T_h is defined by the relation $(T_h u)(x)=u(x-h)$. We are concerned with *explicit*, k-step finite difference schemes of the form

$$U_{n+k}=\sum_{l=0}^{k-1}\sum_{j=-m}^{m} a_{lj}T_h^j U_{n+l}, \qquad k\leqq n+k\leqq T^*/\tau, \tag{2.1}$$

where U_n, $n=0,1,\dots[T/h]$ are complex functions of the real variable x, $-\infty<x<\infty$ and the coefficients a_{lj} depend *only* on the ratio r. What follows can easily be extended to the cases of several space variables and vector valued U_n (the coefficients a_{lj} are then matrices). It is also possible (but not so easy) to let the coefficients a_{lj} depend on τ, h individually, rather than through the combination r alone. For simplicity, none of these extensions is considered here. We have intentionally failed to mention the PDE that (2.1) is meant to approximate, as that information plays no role in stability considerations, these depending on the difference scheme alone.

It is useful to reformulate (2.1) as a one-step recursion

$$\begin{pmatrix} U_{n+k} \\ \vdots \\ U_{n+1}\end{pmatrix} = C(h)\begin{pmatrix} U_{n+k-1} \\ \vdots \\ U_n\end{pmatrix}, \tag{2.2}$$

$$C(h)=\begin{pmatrix} \sum_j a_{k-1,j}T_h^j & \dots & \sum_j a_{1j}T_h^j & \sum_j a_{0j}T_h^j \\ I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & 0 \end{pmatrix} \tag{2.3}$$

where I stands for the identity operator.

Let $H^s=H^s(\mathbb{R})$ denote the usual Sobolev space $s\geqq 0$, $H^0=L^2$ and let us use the notation $(H^s)^k$ for the product of k copies of H^s.

Definition 1. The scheme (2.1) is said to be *weakly stable* (*with index* $s\geqq 0$) if there exist positive constants K, h_0 such that for $0<h\leqq h_0$, $(n+k)\tau\leqq T^*$

$$\|C(h)^n\|\leqq Kh^{-s}, \tag{2.4}$$

where $\|\cdot\|$ denotes the operator norm on $(L^2)^k$.

Definition 2. The scheme (2.1) is said to be (H^s, L^2)-*stable* ($s\geqq 0$) if there exist positive constants K, h_0 such that for $0<h\leqq h_0$, $(n+k)\tau\leqq T^*$

$$\|C(h)^n\|^*\leqq K, \tag{2.5}$$

where $\|\cdot\|^*$ denotes the norm in the space of bounded operators $\mathscr{L}((H^s)^k,(L^2)^k)$.

Upon introducing the Fourier transform with dual variable (or frequency variable) ξ, formulae (2.2)–(2.5) become respectively [14]:

$$\begin{pmatrix} \hat{U}_{n+k}(\xi) \\ \vdots \\ \hat{U}_{n+1}(\xi) \end{pmatrix} = \hat{C}(h\xi)\begin{pmatrix} \hat{U}_{n+k-1}(\xi) \\ \vdots \\ \hat{U}_n(\xi) \end{pmatrix}, \tag{2.6}$$

$$\hat{C}(\theta)=\begin{pmatrix} \sum_j a_{k-1,j}e^{ij\theta} & \dots & \sum_j a_{1j}e^{ij\theta} & \sum_j a_{0j}e^{ij\theta} \\ 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \tag{2.7}$$

$$\sup_\theta |\hat{C}(\theta)^n|\leqq Kh^{-s}, \tag{2.8}$$

$$\sup_\theta \{|\hat{C}(\theta)^n|(1+|\theta h^{-1}|^2)^{-s/2}\}\leqq K. \tag{2.9}$$

In (2.8), (2.9), $|\hat{C}(\theta)^n|$ represents the spectral norm of the $k\times k$ matrix $\hat{C}(\theta)^n$ and the supremum is taken over all real θ (or over $|\theta|\leqq\pi$ as $\hat{C}(\theta)$ is 2π-periodic).

Definition 3. The scheme (2.1) is said to be 0-stable if there exists a positive constant K such that for all n

$$|\hat{C}(0)^n|\leqq K. \tag{2.10}$$

Remark. The term 0-stable is borrowed from the field of numerical ODEs. Note that setting $\theta=0$ in $\hat{C}(\theta)$ means that we are dealing with the frequency ξ

$=0$, i.e. with *constant* (x-independent) U_n. For constant functions (2.1) reduces to a linear, ordinary difference equation with characteristic polynomial

$$P_0(\zeta)=\zeta^k-(\sum_j a_{k-1,j})\zeta^{k-1}-\ldots-(\sum_j a_{0j}).$$

The matrix $\hat{C}(0)$ is the companion matrix of $P_0(\zeta)$ and therefore the zeros of $P_0(\zeta)$ are the eigenvalues of $\hat{C}(0)$. Thus (2.10) is equivalent to the requirement that $P_0(\zeta)$ has all its zeros in the closed unit disk and the zeros with modulus one are simple [1, p. 173].

Theorem 1. *The scheme* (2.1) *is weakly stable if and only if for each real θ all the eigenvalues of $\hat{C}(\theta)$ are in the closed unit disk. Furthermore, if* (2.1) *is weakly stable then the index s can be chosen to be $k-1$.*

Proof. The "only" part is clear, since an eigenvalue in $|\zeta|>1$ would mean an exponential growth of the powers of $\hat{C}(\theta)$. The rest of the theorem is direct application of an algebraic lemma ([14], Lemma 3.3).

Remark 1. The requirement that $\hat{C}(\theta)$ has no eigenvalue in $|\zeta|>1$ is the celebrated *von Neumann* condition. This demands that the eigenvalues are bounded by $1+K\tau$, but in the present circumstances $C(\theta)$ does not depend on τ, h (since the coefficients a_{lj} do not) and therefore K can be chosen to be zero. The equivalence between weak stability and the von Neumann condition was proved by Kreiss in his celebrated paper [6]. His proof, which applies to the $a_{lj}=a_{lj}(\tau,h)$ case, is more involved than that of our Theorem 1, restricted to the situation $a_{lj}=a_{lj}(r)$.

Remark 2. Again we note that the eigenvalues of $\hat{C}(\theta)$ are the zeros of its characteristic polynomial

$$P_\theta(\zeta)=\zeta^k-(\sum_j a_{k-1,j}e^{ij\theta})\zeta^{k-1}-\ldots-(\sum_j a_{0j}e^{ij\theta}).$$

Theorem 2. *The scheme* (2.1) *is (H^s,L^2)-stable if and only if it is* 0*-stable and weakly stable with index s.*

Proof. We first deal with the "only if" part. If (2.9) holds for $(n+k)\tau\leqq T^*$, $h\leqq h_0$ then for $|\theta|\leqq\pi$ and $h\leqq\min(h_0,\pi)$

$$|\hat{C}(\theta)^n|\leqq K(1+(\pi h^{-1})^2)^{s/2}\leqq K 2^{s/2}\pi^s h^{-s}$$

so that (2.8) holds. It is quite clear that (2.9) implies (2.10).

We now turn to the "if" part. The bound (2.8) ensures, according to Theorem 1, that for each θ the eigenvalues of $C(\theta)$ are in the closed unit disk. By continuity, (2.10) implies that there exists $\varepsilon>0$ such that for $|\theta|\leqq\varepsilon$, $\hat{C}(\theta)$ has no multiple eigenvalue on the unit circle $|\zeta|=1$. A well-known result (see e.g. [3] Chapter 11) shows then that the resolvent $(\hat{C}(\theta)-\zeta I)^{-1}$ is regular in $|\zeta|>1$ (including $\zeta=\infty$) and does not possess multiple poles on $|\zeta|\geqq 1$. Then, for each fixed θ, $|\theta|\leqq\varepsilon$

$$f(\zeta,\theta)=(|\zeta|-1)|(\hat{C}(\theta)-\zeta I)^{-1}|$$

is bounded as ζ varies in $|\zeta|\geqq 1$ (including $\zeta=\infty$). Furthermore $f(\zeta,\theta)$ is continuous except at the simple poles of the resolvent, where $f(\zeta,\theta)$ equals the modulus of the corresponding residue. Hence $f(\zeta,\theta)$ is bounded for $|\theta|\leqq\varepsilon$, $|\zeta|\geqq 1$.

It follows from the Kreiss matrix theorem [6, 9] that there exists a constant R such that for all n and $|\theta|\leqq\varepsilon$

$$|\hat{C}(\theta)^n|\leqq R. \tag{2.11}$$

For $|\theta|\geqq\varepsilon$, (2.8) shows that, for $h\leqq h_0$, $(n+k)\tau\leqq T^*$

$$|\hat{C}(\theta)^n|\leqq K\varepsilon^{-s}(|\theta|h^{-1})^s. \tag{2.12}$$

Now (2.11)-(2.12) lead easily to (2.9) and the proof is complete.

Remark. Thomee [14] deals with *one-step* methods which are consistent with a well-posed initial value constant coefficient problem. For those methods $\hat{C}(0)=1\in\mathbb{R}$ (or $\hat{C}(0)=I$ in the vector-valued case) and therefore the 0-stability requirement is trivially satisfied. Thus in his context there is no difference between weak stability and (H^s,L^2)-stability and in fact [14] takes our Definition 2 as definition of weak stability. We believe that our use of the term weak stability is in agreement with the standard terminology, see e.g. [1, 9].

3. The CFL Principle

This brief Section is, as the previous one, of an ancillary character and reviews the CFL principle as applied to the model problem

$$\begin{aligned} &u_t=u_x, \quad 0<t\leqq T^*<\infty, \quad -\infty<x<\infty,\\ &u(x,0)=u_0(x), \quad -\infty<x<\infty. \end{aligned} \tag{3.1}$$

The principle is embodied in the following result ($\mathscr{D}$ denotes the class of indefinitely differentiable functions with compact support).

Theorem 3. *Assume that a scheme of the form* (2.1) *with $r=\tau/h$ constant, is applied to the integration of* (3.1) *in such a way that $U_0=u_0$ and the missing starting levels $U_1,\dots,U_{k-1}$ are computed by means of a one-step method of the form* (2.1) *employing the same values of τ and h. Assume that the method is convergent in either of the two following senses:*

(i) *For each $u_0\in\mathscr{D}$*

$$\lim_{\tau\to 0}\ \sup_{0\leqq n\tau\leqq T^*}\|U_n-u(\cdot,n\tau)\|_{L^2}=0. \tag{3.2}$$

(ii) *For each $u_0\in\mathscr{D}$ and each real x, one has as $n\to\infty$, $\tau\to 0$, $n\tau$ fixed*

$$\lim U_n(x)=u(x,n\tau).$$

Then

$$\tau/h\leqq m. \tag{3.3}$$

Proof. The value $U_n(x)$ only depends on the restriction of u_0 to the interval $J=[x-a,x+a]$, $a=mhn$, whereas $u(x,n\tau)=u_0(x+n\tau)$. If $\tau/h>m$ then $x+n\tau\notin J$ and it is possible to choose $u_0\in\mathscr{D}$ such that $u_0\equiv 0$ in a neighbourhood of J and $u_0\equiv 1$ in a neighbourhood of $x+n\tau$. Then, in the neighbourhood of x, $u(\cdot,n\tau)\equiv 1$, $U_n(\cdot)\equiv 0$ and convergence cannot take place.

Remark. The derivation of the restriction (3.3) on the mesh-ratio r is peculiar in that it stems from a *convergence* requirement via a geometric argument, while conditions on r are usually reached by an algebraic study of the difference scheme on its own, without reference to the PDE being approximated.

4. Disk Like Theorems

To solve initial value problems, we consider "linear" explicit methods [5, 14] which satisfy the following property. When applied to the linear equation $y'=\lambda y$ with a constant step τ the method yields a numerical solution which satisfies a recurrence relation of the form

$$y_{n+k}=\sum_{l=0}^{k-1}\sum_{j=0}^{m} b_{lj}\tau^j\lambda^j y_{n+l}, \qquad n=0,1,2,\ldots \tag{4.1}$$

where b_{lj} are real numbers depending only on the method. The integers k and m are called the number of steps and the number of stages, respectively. Most practical methods, including linear multistep and Runge-Kutta schemes, are of the form (4.1) ([5, 14]). The unknown function y can take values in $\mathbb{C}$, $\mathbb{C}^d$ or even in a functional space X and λ must be interpreted accordingly as a complex number, a complex $d\times d$ matrix or a linear operator on X. The characteristic polynomial associated with (4.1) is, by definition,

$$\Phi(\zeta,\mu)=\zeta^k-\sum_{l=0}^{k-1}\sum_{j=0}^{m} b_{lj}\mu^j\zeta^l \tag{4.2}$$

and the *stability region* S of the method is the set of complex numbers μ for which the roots ζ_l of $\Phi(\zeta,\mu)=0$ are in the closed unit disk, those roots with modulus unity being simple. It is convenient to introduce the *enlarged stability region* S^* of the method, the set of those μ for which the roots ζ_l of $\Phi(\zeta,\mu)=0$ are in the closed unit disk. Clearly S^* is closed and therefore contains the closure $\bar{S}$ of S. The example $m=0$, $k=2$, $b_{10}=2$, $b_{00}=-1$ has $S=\bar{S}=\emptyset$ and $S^*=\mathbb{C}$, so that S^* may be much larger than $\bar{S}$. However for the methods used in practice one often has $S^*=\bar{S}$ (cf. Remark 2 after Theorem 5 below). The method is said to be 0-stable if $0\in S$.

Hereafter we assume that the methods considered are *consistent*, i.e. that the algebraic function $\zeta(\mu)$ defined by (4.2) has a branch $\zeta_1(\mu)$ which is analytic in the neighbourhood of $\mu=0$ and satisfies

$$\zeta_1(\mu)-\exp(\mu)=O(\mu^2), \qquad \mu\to 0.$$

If $r>0$, $\beta\in\mathbb{R}$ we consider in the complex μ-plane the "ellipse" $E(\beta,r)$ with parametric equation

$$\mu(\theta)=r\,\beta(\cos\theta-1)+i\,r\sin\theta, \qquad -\pi\leqq\theta\leqq\pi.$$

When $\beta=0$, $E(\beta,r)$ degenerates in the *interval* $\{i\,y{:}\,y\in[-r,r]\}$. When $\beta\neq 0$, $E(\beta,r)$ is an ellipse with centre at the point $-\beta r$ of the complex plane and axes parallel to the coordinate axes. The horizontal and vertical semi-axes have lengths $|\beta r|, r$ respectively. Therefore $E(1,r)$ is a *circle*.

Our aim in this paragraph is to ascertain whether it is possible for a method to have $E(\beta,r)\subset S^*$. To this end we follow an indirect approach. We consider the PDE (3.1) and discretize it in space as follows

$$U_t(x,t)=(2h)^{-1}\left[(\beta+1)\,T_h-2\beta I+(\beta-1)\,T_h^{-1}\right]U(x,t). \tag{4.3}$$

Here the translations refer to the space variable x. The case $\beta=0$ corresponds to central differences and the case $\beta=1$ to backward differences. Now (4.3) can be viewed as an abstract Cauchy problem $U_t=h^{-1}AU$ which can in turn be discretized in time by means of (4.1) with step-size $\tau=rh$ to yield

$$U_{n+k}=\sum_{l=0}^{k-1}\sum_{j=0}^{m} b_{lj}\,r^j A^j\,U_{n+l}, \tag{4.4}$$

with

$$2A=(\beta+1)\,T_h-2\beta I+(\beta-1)\,T_h^{-1}.$$

Clearly (4.4) is a consistent method of the form (2.1) for the numerical integration of (3.1). (For instance, when (4.1) is chosen to be Euler's rule and $r=\beta$, the method (4.4) is the Lax-Wendroff scheme).

The cornerstone of this paper is the following simple result.

Theorem 4. *The ellipse $E(\beta,r)$ is contained in the enlarged stability region S^* of the* ODE *method* (4.1) *if and only if the* PDE *method* (4.4) *is weakly stable.*

Proof. According to Theorem 1 and to the remarks which follow it, weak stability is equivalent to the requirement that for $-\pi\leqq\theta\leqq\pi$ the roots of

$$\zeta^k=\sum_{l=0}^{k-1}\sum_{j=0}^{m} b_{lj}\,r^j\hat{A}^j\zeta^l,$$

with

$$\begin{aligned}\hat{A}&=(1/2)\left[(\beta+1)\,e^{i\theta}-2\beta+(\beta-1)\,e^{-i\theta}\right]\\&=\beta(\cos\theta-1)+i\sin\theta,\end{aligned}$$

belong to the closed unit disk, so that the result follows trivially from the definition of S^*.

Remark. The maximum principle on Riemann surfaces shows that $E(\beta,r)\subset S^*$ if and only if S^* contains $E(\beta,r)$ together with its "interior" (i.e. the bounded component of $\mathbb{C}\setminus E(\beta,r)$).

A first consequence of Theorem 4 is the following generalization of the results (A) and (B) of the introduction, first obtained by Jeltsch and Nevanlinna.

Theorem 5. *Assume that* (4.1) *is consistent and* 0-*stable and that the ellipse* $E(\beta, r)$, $\beta \geqq 0$ *is contained in the enlarged stability region* S^*. *Then* $r \leqq m$.

Proof. The 0-stability of (4.1) (in the sense of the definition given in this Section) clearly forces the 0-stability of the PDE scheme (4.4) in the sense of Sect. 2. Theorem 4 guarantees that (4.4) is also weakly stable and therefore (Theorem 2) (H^s, L^2)-stable. Standard results (see e.g. [14]) show that the approximations generated by (4.4) converge in the L^2-norm towards the theoretical solution, whenever the initial datum u_0 is in H^s and the starting values $U_0, \ldots, U_{k-1}$ are all equal to u_0. According to Theorem 3 (the CFL principle) $r \leqq m$.

Remark 1. As noted before, the cases $\beta = 1, 0$ correspond respectively to the results (A), (B) quoted in the introduction. The case of a general β, not considered in the original work of Jeltsch and Nevanlinna can also be proved by means of the methods in their paper [5]. One would have to apply Theorem 2.5 of [5], with Q given by $k=2$, $m=1$

$$b_{10} = 2\beta/(1+\beta), \quad b_{00} = (1-\beta)/(1+\beta),$$
$$b_{11} = 2/(1+\beta), \quad b_{01} = 0.$$

Remark 2. Our result is slightly stronger than those stated in [5] in that we do not assume $\Phi(\zeta, \mu)$ to be irreducible and we use the larger set S^* rather than S or $\bar{S}$. However the proofs in [5] would also work under these less demanding hypotheses. Also note that $\Phi(\zeta, \mu)$ is invariably irreducible for practical methods and that, in the irreducible case, $E(\beta, r) \subset S^*$ if and only if $E(\beta, r) \subset \bar{S}$. In fact, suppose that $\Phi(\zeta, \mu)$ is irreducible and that there exists a point $\mu_0 \in E(\beta, r) \subset S^*$ such that $\mu_0 \notin \bar{S}$. Then $\mu \notin \bar{S}$ if $\mu \in E(\beta, r)$ and μ is near μ_0. For those values of μ, $\Phi(\zeta, \mu)$ has a multiple root and this contradicts the hypothesis of irreducibility.

Remark 3. It should be pointed out that the technique in [5], related to the powerful order-star concept [15], is capable of characterizing the methods for which the equality $r = m$ can be attained (cf. Sect. 6).

Remark 4. Theorem 4 is also valid for negative β. However no consistent, explicit method can have $E(\beta, r) \subset S^*$ if $\beta < 0$. This is a simple consequence of the remark after Theorem 4, since consistency implies that for μ positive and small $\mu \notin S^*$.

5. Algebraic Proofs of CFL Bounds

So far, a geometric argument has led us to the CFL principle (Theorem 3) and then the bound (3.3), given by that principle, has been used in the proof of

Theorem 5 concerning regions of stability. In this section we show that the statement in Theorem 5 implies that in Theorem 3, so that these two theorems can be regarded as equivalent. Since we know that the validity of Theorem 5 can be proved by the algebraic method of Jeltsch and Nevanlinna, we conclude that it is possible to establish the CFL bound (3.3) by purely algebraic techniques.

In this Section, and for simplicity, we restrict our attention to one-step methods ($k=1$) and set $U_0=u_0$. We assume that Theorem 5 has been proved à la Jeltsch-Nevanlinna and we wish to show that $\tau/h \leqq m$ for a consistent method of the form (4.4) which converges in one of the sense (i)–(ii) considered in Theorem 3. After Theorem 4, it is enough to prove that "convergent" methods are weakly stable.

We first consider convergence in the sense of (i). We have not been able to show that this convergence implies weak stability. The usual Lax-Richtmyer equivalence theorem [9], as generalized in [11], does not cater for the present situation since $\mathscr{D}$ with the H^s topology is not a Banach space, and the completeness hypothesis is *essential* in this sort of Theorem [7, 8]. Our best result for this type of convergence is as follows.

Theorem 6. *Assume that $k=1$ and that for the approximations given by the PDE method* (4.4), *with $U_0=u_0$, the limit* (3.2) *holds for each $u_0 \in H^\infty = \bigcap_{s\in N} H^s$. Then* (4.4) *is (L^2, L^2)-stable (i.e. Lax-stable).*

Proof. The space H^∞ is a Frechet space, when endowed simultaneously with all the H^s-norms. Thus H^∞ is a barelled space [10]. According to an extension of the Lax equivalence theorem due to Schultz [12], the family of powers $C(h)^n$, $0 \leqq n\tau \leqq T^*$, $0 < h \leqq h_0$ is an equicontinuous family of linear operators from H^∞ into L^2. Therefore [10] there exists a nonnegative s such that in the $\mathscr{L}(H^s, L^2)$-norm (2.5) holds. Thus (4.4) is (H^s, L^2)-stable and, by Theorem 2, weakly stable. Theorem 1 shows finally that s can be chosen to be zero.

Remark. If the convergence is demanded not only when U_0 is taken to be u_0, but also when U_0 tends to u_0 as $\tau \to 0$ then a method is stable if it is convergent for the trivial initial datum $u_0 \equiv 0$ ([1, 7, 11]).

We now turn to the convergence in the pointwise sense (ii).

Theorem 7. *Assume that $k=1$ and that, when $U_0=u_0$, the method* (4.4) *is convergent in the pointwise sense of* (ii), *Theorem* 3. *Then* (4.4) *is (L^2, L^2)-stable.*

Proof. We take in (ii), $n\tau = T^*$ and $x=0$. For each nonnegative integer n and function ϕ, we define $\psi_n(\phi)$ to be the value at $x=0$ of the numerical solution generated by n steps of (4.4) with $\tau = T^*/n$ and $U_0=\phi$. More precisely

$$\psi_n(\phi) = [C^n(h)\,\phi](0), \qquad n\tau = T^*, \qquad \tau/h = r.$$

By hypothesis

$$\lim_{n\to\infty} \psi_n(\phi) = \phi(T^*), \qquad \phi \in \mathscr{D}.$$

We next introduce the space $\mathscr{B}$ of bounded real functions with bounded derivatives of all orders and endow $\mathscr{B}$ with the family of norms

$$\|\phi\|_s = \sum_{j=0}^{s} \sup_{x\in R} |\phi^{(j)}(x)|, \qquad s=0,1,2,\ldots.$$

Take $\rho\in\mathscr{D}$ such that $\rho(T^*)=1$ and that $\rho(x)=1$ for x in the numerical domain of dependence of the point $x=0$, $t=T^*$. Then for $\phi\in\mathscr{B}$ the product $\rho\phi$ is in $\mathscr{D}$ and

$$\lim \psi_n(\phi) = \lim \psi_n(\rho\,\phi) = \rho(T^*)\,\phi(T^*) = \phi(T^*)$$

i.e. pointwise convergence holds also for all initial datum in $\mathscr{B}$. Thus $\{\psi_n\}$ is a pointwise convergent family of linear functionals on the barelled space $\mathscr{B}$. Clearly each ψ_n is continuous and therefore [10] $\{\psi_n\}$ is an equicontinuous family. Hence an integer s and a constant K can be found, so that for all $\phi\in\mathscr{B}$, $n\in\mathbb{N}$

$$|\psi_n(\phi)| \leqq K\,\|\phi\|_s.$$

The particular choice $\phi_\theta(x)=\exp(i\theta h^{-1} x)$ leads to

$$|\psi_n(\phi_\theta)| \leqq K \sum_{l=0}^{s} |\theta h^{-1}|^l. \tag{5.1}$$

Now we have the equality of functions of x

$$C^n(h)\,\phi_\theta = \hat{C}(\theta)^n\,\phi_\theta. \tag{5.2}$$

(Note that both sides of (5.2) have the same Fourier transform.) ((5.2) is the "naive" definition of amplification factor often found in elementary books.) Evaluation of (5.2) at the point $x=0$ leads to $\psi_n(\phi_\theta) = \hat{C}(\theta)^n \cdot 1$. Upon substituting this last equality into (5.1) we conclude that the scheme is (H^s, L^2)-stable.

Remark. It is clear that the idea behind this proof possesses a wide range of applicability, not limited to hyperbolic cases.

6. Schemes With Optimal Stability Properties

For a method of the form (4.4), the CFL condition is *necessary* for weak stability. The techniques used in [5] to prove Theorem 5.1, employed in combination with our Theorem 4, lead easily to the following result, which characterizes the methods for which the CFL condition is also sufficient for weak stability.

Theorem 8. *Let the method* (4.4) *be consistent and assume that the method* (4.1) *used to perform the time-stepping is irreducible. Then* (4.4) *is weakly stable at* $r=m$ *if and only if* (4.4) *is given by*

$$\Phi(\zeta,\mu) = \zeta^2 - \frac{2}{(1+\beta)^m} \sum_{j=0}^{[m/2]} \binom{m}{2j} \left(\beta + \frac{\mu}{m}\right)^{m-2j} \left(\frac{\mu^2}{m^2} + \frac{2\beta\mu}{m} + 1\right) \zeta + \left(\frac{\beta-1}{\beta+1}\right)^m;$$

$\beta \neq 1$

or

$$\Phi(\zeta, \mu) = \zeta - (1 + \mu/m)^m, \qquad \beta = 1.$$

As an example of the application of the Theorem suppose that $u_t = u_x$ is discretized by forward differences $\beta = 1$. Then either the time-stepping is carried by means of Euler's rule or the fully discrete scheme cannot employ the maximum τ allowed by the CFL condition. (Note that one step of length τ of the method $(1 + \mu/m)^m$ is tantamount to m consecutive steps of Euler's rule with step τ/m.)

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