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A HAMILTONIAN, EXPLICIT ALGORITHM WITH SPECTRAL ACCURACY FOR THE 'GOOD' BOUSSINESQ SYSTEM

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We construct an explicit pseudo-spectral method for the numerical solution of the soliton-producing 'good' Boussinesq system $w_t = u_{xxx} + u_x + (u^2)_x$, $u_t = w_x$. The new scheme preserves a discrete Poisson structure similar to that of the continuous system. The scheme is shown to converge with spectral spatial accuracy. A numerical illustration is given.

1. Introduction

Physical systems are often described by a set of ordinary or partial differential equations in Hamiltonian form. The discretization of such systems by algorithms that preserve the Hamiltonian structure has recently been the subject of several contributions, see e.g. [1–7]. The Leit Motiv associated with such discretizations is that they automatically inherit many qualitative features of the continuous system. In this paper we suggest a Hamiltonian, time-discrete, pseudo-spectral scheme for the soliton producing 'good' Boussinesq system [8]:

$$w_t = -u_{xxx} + u_x + (u^2)_x\,, \qquad u_t = w_x\,. \tag{1.1}$$

The new scheme is derived in Section 2. Its nonlinear stability and convergence are proved in Section 3.

A different time-discrete pseudo-spectral scheme has been analysed by the present authors in [9]. That paper and [8] contain further references concerning (1.1) and its numerical solution.

2. Hamiltonian discretization

We study the 1-periodic problem for (1.1) on $0 \leq t \leq T < \infty$, with initial conditions

$$w(x,0) = w^0(x)\,, \qquad u(x,0) = u^0(x)\,. \tag{2.1}$$

We start by rewriting (1.1) as

$$(\mathrm{d}/\mathrm{d}t)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix} \begin{bmatrix} w \\ -u_{xx} + u + u^2 \end{bmatrix} . \tag{2.2}$$

If the letter z refers to the vector $[w, u]^{\mathrm{t}}$, m denotes the matrix operator on the right-hand side of (2.2) and δ denotes the variational derivative, then (2.2) becomes

$$z_t = m\,\delta h(z) , \tag{2.3}$$

where

$$h(z) = \int_0^1 (\tfrac{1}{2}w^2 + \tfrac{1}{2}(u_x)^2 + \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3)\,\mathrm{d}x \tag{2.4}$$

is the Hamiltonian. The system (2.3) is in the so-called Poisson form [5, 6, 10], a generalization of the form of the familiar Hamilton equations. The operator m is skew-symmetric and defines the Poisson bracket $\{F, G\}$ of pairs of functionals $F(z)$, $G(z)$ via $\{F, G\} = (\delta F, m\,\delta h)$, where $(\cdot, \cdot)$ denotes the L^2-inner product. In turn, the Poisson bracket defines a Poisson structure [5, 6, 10] which is preserved by the flow of (2.3). Furthermore (2.4) is a constant of motion because

$$\mathrm{d}h/\mathrm{d}t = (\delta h, \mathrm{d}z/\mathrm{d}t) = (\delta h, m\,\delta h) = 0 . \tag{2.5}$$

Of importance is that (2.5) holds in view of the structure of (2.3) and the skew-symmetry of m; the actual form of m is of no consequence.

We now describe the Hamiltonian space-discretization of (1.1). If J is a positive integer, we set $h = 1/(2J)$ and consider the mesh $\{x_j = jh \mid j \text{ an integer}\}$. We denote by $\mathbb{Z}_h$ the space of real 1-periodic functions defined on the mesh. Thus each element $V \in \mathbb{Z}_h$ is a sequence $\{V_j\}_{j=0,\pm 1,\dots}$ with $V_j = V_{j+2J}$, $j = 0, \pm 1, \dots$ We denote by D the standard pseudo-spectral [11] discretization of ∂_x and by $(\cdot, \cdot)$ and $\|\cdot\|$ the L^2-inner product and norm in $\mathbb{Z}_h$. Then we can construct the following discrete Hamiltonian, cf. (2.4),

$$H = \tfrac{1}{2}(W, W) + \tfrac{1}{2}(\mathrm{D}U, \mathrm{D}U) + \tfrac{1}{2}(U, U) + \tfrac{1}{3}(U^2, U) , \quad W, U \in \mathbb{Z}_h , \tag{2.6}$$

and the skew-symmetric operator in $\mathbb{Z}_h \times \mathbb{Z}_h$ given by

$$M = \begin{bmatrix} 0 & \mathrm{D} \\ \mathrm{D} & 0 \end{bmatrix} . \tag{2.7}$$

With $Z = [W, U]^{\mathrm{t}}$ the suggested space-discretization of (2.3) is

$$Z_t = M\,\delta H(Z) . \tag{2.8}$$

Just as m in (2.3), M in (2.7) defines a Poisson structure, which is preserved by the flow of (2.8). Furthermore H is a constant of motion. The proof in (2.5) is still valid replacing h, z, m

by H, Z, M and understanding δ as meaning gradient. When written in full, (2.8) becomes, cf. (1.11),

$$\boldsymbol{W}_t = -\mathrm{D}^3\boldsymbol{U} + \mathrm{D}\boldsymbol{U} + \mathrm{D}\boldsymbol{U}^2\,, \quad \boldsymbol{U}_t = \mathrm{D}\boldsymbol{W}\,. \tag{2.9}$$

Now (2.9) is discretized in time by the following staggered, explicit algorithm:

$$(\boldsymbol{W}^{n+1/2} - \boldsymbol{W}^{n-1/2})/\Delta t = -\mathrm{D}^3\boldsymbol{U}^n + \mathrm{D}\boldsymbol{U}^n + \mathrm{D}(\boldsymbol{U}^n)^2\,,$$

$$(\boldsymbol{U}^{n+1} - \boldsymbol{U}^n)/\Delta t = -\mathrm{D}\boldsymbol{W}^{n+1/2}\,. \tag{2.10}$$

By using the characterization in formula (5) of [6], it is an exercise to show that the map $[\boldsymbol{W}^{n-1/2}, \boldsymbol{U}^n] \rightarrow [\boldsymbol{W}^{n+1/2}, \boldsymbol{U}^{n+1}]$ is a Poisson map, i.e. (2.10) preserves the Poisson structure associated with (2.7). However (2.10) does not conserve the energy (2.6) and in fact a general result by Zhong and Marsden [7] shows that, in the time-integration, it is not possible to conserve both the Hamiltonian and the Poisson structure. See also the discussion in [4, Section 3].

The scheme (2.10) is supplemented by the initial conditions

$$\boldsymbol{U}^0 = \boldsymbol{\alpha}\,, \qquad \boldsymbol{W}^{1/2} = \boldsymbol{\beta}\,, \tag{2.11}$$

with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ approximations to the grid restrictions $\boldsymbol{r}_h u^0$, $\boldsymbol{r}_h w^{1/2}$ of $u(\cdot, 0)$, $w(\cdot, \frac{1}{2}\Delta t)$.

For implementation purposes (2.10) is Fourier transformed, so that the time-stepping is performed in Fourier space. Then, the method requires, per step, an inverse Fourier transform to recover $\boldsymbol{U}$ from $\mathcal{F}\boldsymbol{U}$ before forming $\boldsymbol{U}^2$ and a Fourier transform to compute $\mathcal{F}\boldsymbol{U}^2$. Furthermore an extra inverse transform is needed whenever one wishes to get output $\boldsymbol{W}$.

3. Nonlinear stability and convergence

In what follows h and Δt are varied according to $\Delta t/h^2 = r$, r a constant $<2/\pi^2$. For pairs $[\boldsymbol{W}, \boldsymbol{U}] \in \mathbb{Z}_h \times \mathbb{Z}_h$ we use the energy norm

$$\|[\boldsymbol{W}, \boldsymbol{U}]\|_{\mathrm{E}^2} = \|\mathrm{D}^{-1}\boldsymbol{W}\|^2 - rh^2(\mathrm{D}\boldsymbol{W}, \boldsymbol{U}) + \|\boldsymbol{U}\|^2\,, \tag{3.1}$$

where D^{-1} is the operator such that $\mathrm{D}^{-1}\mathrm{D}\boldsymbol{V} \equiv \boldsymbol{V} - \langle \boldsymbol{V} \rangle$; $\langle \cdot \rangle$ denotes the mean value. The existence and uniqueness of D^{-1} is easily established by means of Fourier analysis. Since D has eigenvalues $2\pi ij$, $|j| \leq J$,

$$rh^2(\mathrm{D}\boldsymbol{W}, \boldsymbol{U}) \leq rh^2\|\mathrm{D}\boldsymbol{W}\|\,\|\boldsymbol{U}\| \leq rh^2 4\pi^2 J^2 \|\mathrm{D}^{-1}\boldsymbol{W}\|\,\|\boldsymbol{U}\| = r\pi^2\|\mathrm{D}^{-1}\boldsymbol{W}\|\,\|\boldsymbol{U}\|$$

and, since we have assumed $r\pi^2 < 2$ (3.1) defines in fact a norm. Furthermore, for each fixed $r < 2/\pi^2$, the norm $\|[\boldsymbol{W}, \boldsymbol{U}]\|_{\mathrm{E}}$ is equivalent, uniformly in h, to the Sobolev norm

$$(\|\mathrm{D}^{-1}\boldsymbol{W}\|^2 + \|\boldsymbol{U}\|^2)^{1/2}\,. \tag{3.2}$$

To study the nonlinear stability of (2.10), let $\{W^{n-1/2}\}_{1\leq n\leq N}$, $\{U^n\}_{1\leq n\leq N}$, $N=[T/\Delta t]$ be sequences of elements of $\mathbb{Z}_h$, not necessarily satisfying (2.10), and define the residuals

$$R^{n+1/2}=(W^{n+1/2}-W^{n-1/2})/\Delta t+\mathrm{D}^3U^n-\mathrm{D}U^n-\mathrm{D}(U^n)^2\,,\quad n=1,2,\ldots,N-1\,,$$

$$S^{n+1}=(U^{n+1}-U^n)/\Delta t-\mathrm{D}W^{n+1/2}\,,\quad n=1,2,\ldots,N-1\,.$$

Furthermore let $\{W^{*n-1/2}\}_{1\leq n\leq N}$, $\{U^{*n}\}_{1\leq n\leq N}$, be another couple of sequences with corresponding residuals $\{R^{*n+1/2}\}_{1\leq n\leq N-1}$, $\{S^{*n+1}\}_{1\leq n\leq N-1}$. We then have Theorem 1.

THEOREM 1. Assume that $\Delta t/(h^2)=r<2/\pi^2$ and that (1.1), (2.1) possesses a bounded solution (w,u). Set $M=\max\{|u|\mid 0\leq x\leq 1,\ 0\leq t\leq T\}$ and let μ be an arbitrary positive number. Then there exists a constant C depending only on r, M, μ, T, such that if

$$\max_{1\leq n\leq N}\|U^n-r_hu^n\|\leq\mu h^{1/2}\,,\tag{3.3}$$

$$\max_{1\leq n\leq N}\|U^{*n}-r_hu^n\|\leq\mu h^{1/2}\,,\tag{3.4}$$

then

$$\begin{aligned}\max_{1\leq n\leq N}&\|[W^{n-1/2}-W^{*n-1/2},U^n-U^{*n}]\|_{\mathrm{E}}\\&\leq C\{\|[W^{1/2}-W^{*1/2},U^1-U^{*1}]\|_{\mathrm{E}}\\&\quad+\sum_{1\leq n\leq N-1}\Delta t(\|\mathrm{D}^{-1}(R^{n+1/2}-R^{*n+1/2})\|+\|S^{n+1}-S^{*n+1}\|)\}\,.\end{aligned}\tag{3.5}$$

PROOF. Set the abbreviations $E^{n+1/2}=W^{n+1/2}-W^{*n+1/2}$, $F^{n+1}=U^{n+1}-U^{*n+1}$, $n=0,\ldots,N-1$; $L^{n+1/2}=R^{n+1/2}-R^{*n+1/2}$, $M^{n+1}=S^{n+1}-S^{*n+1}$, $n=1,2,\ldots,N-1$. Subtract the definitions of $R^{n+1/2}$ and $R^{*n+1/2}$ to get, for $n=0,1,\ldots,N-1$,

$$(E^{n+1/2}-E^{n-1/2})/\Delta t=-\mathrm{D}^3F^n+\mathrm{D}F^n+\mathrm{D}\{(U^n)^2-(U^{*n})^2\}+L^{n+1/2}\,,$$

which leads to

$$\begin{aligned}(\mathrm{D}^{-1}E^{n+1/2}-\mathrm{D}^{-1}E^{n-1/2})/\Delta t&=-\mathrm{D}^2F^n+F^n-\langle F^n\rangle\\&\quad+\{(U^n)^2-(U^{*n})^2\}-\langle\{(U^n)^2-(U^{*n})^2\}\rangle+\mathrm{D}^{-1}L^{n+1/2}\,.\end{aligned}\tag{3.6}$$

On the other hand, the subtraction of the definitions of S^{n+1}, S^{*n+1} yields for $n=1,2,\ldots,N-1$,

$$(F^{n+1}-F^n)/\Delta t=\mathrm{D}E^{n+1/2}+M^{n+1}\,.\tag{3.7}$$

Take the inner product of (3.6) with $\mathrm{D}^{-1}E^{n+1/2}+\mathrm{D}^{-1}E^{n-1/2}$ and that of (3.7) and $F^{n+1}+F^n$

and add to obtain after manipulation (cf. (3.1))

$$\begin{aligned}(\|[E^{n+1/2}, F^{n+1}]\|_E^2 - \|[E^{n-1/2}, F^n]\|_E^2)/\Delta t \\ =(F^n - \langle F^n\rangle, D^{-1}E^{n+1/2} + D^{-1}E^{n-1/2}) \\ +((U^n)^2 - (U^{*n})^2 - \langle\{(U^n)^2 - (U^{*n})^2\}\rangle, D^{-1}E^{n+1/2} + D^{-1}E^{n-1/2}) \\ +(D^{-1}L^{n+1/2}, D^{-1}E^{n+1/2} + D^{-1}E^{n-1/2}) + (M^{n+1}, F^{n+1} + F^n)\,. \end{aligned} \tag{3.8}$$

Note that the nonlinear term in (3.8) can be bounded:

$$\|(U^n)^2 - (U^{*n})^2 - \langle(U^n)^2 - (U^{*n})^2\rangle\|^2 \leq \|U^n - U^{*n}\|^2 \leq \|U^n + U^{*n}\|_\infty \|F^n\|$$

with

$$\begin{aligned}\|U^n + U^{*n}\|_\infty &\leq \|U^n - r_h u^n\|_\infty + \|U^{*n} - r_h u^n\|_\infty + 2\|r_h u^n\|_\infty \\ &\leq 2M + h^{-1/2}\|U^n - r_h u^n\| + h^{-1/2}\|U^{*n} - r_h u^n\| \\ &\leq 2M + 2\mu\,,\end{aligned}$$

in view of (3.3), (3.4). Therefore the proof is easily concluded by applying the Cauchy–Schwarz inequality to each inner product in (3.8) and using the discrete Gronwall lemma. □

To study the convergence of the method we apply Theorem 1 with $\{W^{n-1/2}\}_{1\leq n\leq N}$, $\{U^n\}_{1\leq n\leq N}$, a solution of the recurrence (2.10) and $\{W^{*n-1/2}\}_{1\leq n\leq N}$, $\{U^{*n}\}_{1\leq n\leq N}$, equal to the grid restrictions $\{r_h w^{n-1/2}\}_{1\leq n\leq N}$, $\{r_h u^n\}_{1\leq n\leq N}$. With this choice, the residuals $\{R^{n+1/2}\}_{1\leq n\leq N-1}$, $\{S^{n+1}\}_{1\leq n\leq N-1}$ are zero, while the residuals $\{R^{*n+1/2}\}_{1\leq n\leq N-1}$, $\{S^{*n+1}\}_{1\leq n\leq N-1}$ are truncation errors. For a smooth solution (w, u) of (1.1), these truncation errors are easily seen to be $O(\Delta t^2 + \varphi(h))$, where the function $\varphi(h)$ bounds the spatial part of the truncation error and is spectrally small, i.e., it has bounds of the form $|\varphi(h)| \leq C_s h^s$, for each $s > 0$, or even decreases exponentially with h [11, 12]. Therefore, (3.5) implies

$$\begin{aligned}\max_{1\leq n\leq N} \|[W^{n-1/2} - r_h w^{n-1/2}, U^n - r_h u^n]\|_E \\ = O\{\|[W^{1/2} - r_h w^{1/2}, U^1 - r_h u^1]\|_E + \Delta t^2 + \varphi(h)\}\,.\end{aligned} \tag{3.9}$$

It is a simple matter (cf. Section 4) to choose the starting data $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ in (2.11) to ensure $\|(W^{1/2} - r_h w^{1/2}, U^1 - r_h u^1)\|_E = O(\Delta t^2 + \psi(h))$ with ψ spectrally small, and then (3.9) shows the convergence, in the norms (3.1) or (3.2), of the scheme with spectral accuracy in space and second order accuracy in time. Note that before $\{W^{n-1/2}\}_{1\leq n\leq N}$, $\{U^n\}_{1\leq n\leq N}$ can be substituted into (3.5), condition (3.3) should be checked. However the validity of (3.3) for the numerical $\{W^{n-1/2}\}_{1\leq n\leq N}$, $\{U^n\}_{1\leq n\leq N}$ when h, Δt are small can be ensured as in [9, 13] by the application of an abstract theorem due to López-Marcos and Sanz-Serna [14, 15].

Table 1
Soliton error

$2J$	h	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.025$
32	3.75	0.26E + 0	0.26E + 0	0.26E + 0
64	1.875	0.62E − 2	0.62E − 2	0.62E − 4
128	0.9375	0.38E − 3	0.95E − 4	0.23E − 4

4. Numerical results

Lack of space prevents us from presenting numerical comparisons between (2.10) and other available schemes. The following experiment illustrates the convergence properties discussed in the previous section.

We took $T = 40$ and an intial condition corresponding to a soliton [8] with amplitude $A = 0.5$ located at $x = 0$. Thus the theoretical solution is given by

$$\begin{aligned} u &= 0.5\,\mathrm{sech}^2\left(\tfrac{1}{3}\sqrt{3}\left(x - \tfrac{1}{3}\sqrt{6}t\right)\right), \\ W &= \int_{-\infty}^{x} u_t(\xi, t)\,\mathrm{d}\xi\,. \end{aligned} \tag{4.1}$$

The spatial domain was chosen to be $-60 \leq x \leq 60$ and periodicity conditions were imposed. The functions in (4.1) satisfy these boundary conditions up to a negligible remainder. The starting datum $W^{1/2}$ at $t = \frac{1}{2}\Delta t$ was computed from u, w at $t = 0$, by using the first two terms of the corresponding Taylor expansion.

Table 1 provides the L^2-errors in the u component of the solution (w, u) at the final time. Note that for $2J = 32, 64$ a reduction in Δt does not change the error. Thus, for these values of J the errors originate, almost entirely, from the space-discretization, i.e., the errors given by (2.10) coincide with those of the semidiscretization (2.9). However when $2J = 128$, halving Δt results in a division by 4 of the error, showing that then the spatial discretization is far more accurate than the integration in time. A comparison between the rows of the table nicely shows the spectral spatial accuracy of the scheme.

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