Stochastics and Dynamics, Vol. 8, No. 1 (2008) 47–57

STABILIZING WITH A HAMMER

J. M. SANZ-SERNA

Departamento de Matemática Aplicada,
Universidad de Valladolid, Spain
sanzsern@mac.uva.es

Received 21 November 2007

We show that, with high probability, the undamped, nonlinear inverted pendulum may be "stabilized" over bounded time-intervals if subjected to random shocks.

Keywords: Random impulses; stochastic stabilization.

AMS Subject Classification: 34F05, 60H10, 93E15

1. Introduction

This note presents a simple case-study of the possibility of stabilizing oscillators by means of random impulses.

It is well known that unstable equilibria of linear or nonlinear, damped or undamped oscillators may become stable and even asymptotically stable when the system is subject to a vibrating external forcing term (vibrational stabilization in the terminology of [4]). This fact, oftentimes associated with Kapitza's name, was demonstrated experimentally by Stephenson in 1908 for the case of an inverted pendulum and leads to many useful physical applications that include the Nobel-prize winning Paul trap (see e.g. [9] for a brief historical summary). The case where the external force is harmonic is of course the easiest to analyze and therefore was the first to be studied; the Ince–Strutt diagram arose in this way. The possibility of controlling by vibration unstable oscillators has also received much study, see e.g. Example 12.3.9 of [3] based on [6].

Stochastic forcing terms (stabilization by noise) have been considered in this connection at least since the 60s of the last century (see e.g. the bibliography of [11]). Important contribution are [1, 2]. The literature spawned by the original Stepheson experiment is by now huge and need not be surveyed here; useful references may be obtained from [7, 12].

In this short note, as in [13], the external force is impulsive; more precisely, the (undamped, nonlinear) oscillator is subject to shocks of fixed amplitude that take place after random waiting times. Situations where stochastic perturbations occur via randomly spaced impulses rather than via noise arise in a number of applications [13]. Furthermore, shock models have the advantage that they may be simulated numerically with the integrator of choice for the underlying deterministic oscillator and do not need numerical techniques for SDEs [8]. In molecular dynamics, impulses are used to keep the simulation at a fixed temperature (Andersen bath). For constant energy (microcanical) simulations, it has been suggested [5] to perturb deterministic models by the introduction of noise with a view to turning the simulation "more ergodic" so that the molecule or system of molecules visits a larger fraction of the phase space; random impulses like those considered here may provide a useful way of throwing-in randomness while keeping the deterministic software being used currently.

Our simple model pendulum problem is described in Sec. 2 and analyzed in Sec. 3. It will be shown that, provided that the shocks happen frequently, the inverted pendulum will not fall with arbitrarily high probability over arbitrarily long, finite time-intervals. Section 4 contains some numerical illustrations and the final Sec. 5 provides some concluding observations.

2. An Inverted Pendulum with Random Shocks

Even though more general oscillators could be considered without much extra effort, we limit ourselves to the study of the motion of a pendulum of length ℓ whose suspension point S is subject to a vertical acceleration $a(t)$ with respect to the laboratory. If q measures the angle between the rod and the upward vertical and g denotes the acceleration of gravity, the motion is governed by

$$\frac{d^2q}{dt^2} = \ell^{-1}(g + a(t)) \sin q,$$

or, after introducing the angular velocity p, by

$$\frac{dp}{dt} = \ell^{-1}(g + a(t)) \sin q, \quad \frac{dq}{dt} = p. \tag{2.1}$$

In analogy with [13], we are interested in cases where the acceleration $a(t)$ of S is given formally by a train of Dirac's delta functions or, equivalently, the velocity $v(t)$ of S is a piecewise constant function of t. More precisely, we take

$$a(t) = \sum_{n=1}^{\infty} (-1)^n 2v^* \delta_{t_n}, \tag{2.2}$$

where $v^* > 0$ is a parameter with dimensions of velocity that governs the strength of the shocks and the t_n's are impulse times, $0 < t_1 < t_2 < \dots$, to be described presently. Note that (2.2) corresponds to the acceleration of S if, for $n = 0, 1, 2, \dots$, with $t_0 = 0$,

$$v(t) = v^*, \ t_{2n} < t < t_{2n+1}, \quad v(t) = -v^*, \quad t_{2n+1} < t < t_{2n+2}; \tag{2.3}$$

thus the suspension point moves up and down at a constant speed and reverses its motion at the shock times t_n, $n = 1, 2, \dots$. (This could perhaps be compared with Kac's walk where the reversals occur with probability less than 1.)

Let F be a probability distribution on the half-line $(0, \infty)$ with unit expectation and variance σ^2. The random impulse times t_n depend on a (small) parameter $\tau > 0$ and are derived from F by setting $t_n = t_{n-1} + \tau_n$, $n = 1, 2, \ldots,$ where the random waiting times τ_n are mutually independent and have the common distribution $F(\cdot/\tau)$, so that $\mathbf{E}(\tau_n) = \tau$ and $\mathrm{Var}(\tau_n) = \tau^2\sigma^2$.

Later, simulations will be carried out over finite intervals $0 \le t \le T$. If $N(T)$ denotes the largest index n with $t_n \le T$, it is well known that when the underlying probability distribution F is exponential (leading to the Poisson process for the number of impulses) $\mathbf{E}(\tau N(T)) = T$. For general distributions satisfying the hypotheses above, the same relation holds in the limit:

$$\lim_{\tau \to 0} \mathbf{E}(\tau N(T)) = T. \tag{2.4}$$

(Indeed $\tau N(T)$ is asymptotically normal with mean T and variance $\tau\sigma^2 T$, but this fact will not be used here.) The particular case where F is degenerate and concentrated at 1 yields deterministic impulse times $t_n = n\tau$.

For $a(t)$ given by (2.2), Eqs. (2.1) can be rewritten, without using the Dirac delta function, as

$$\frac{dp}{dt} = \ell^{-1} g \sin q, \quad \frac{dq}{dt} = p, \quad t_{n-1} < t < t_n, \quad n = 1, 2, \ldots, \tag{2.5}$$

$$p(t_n^+) = p(t_n^-) + (-1)^n 2\ell^{-1} v^* \sin q(t_n^-), \quad q(t_n^+) = q(t_n^-); \tag{2.6}$$

in this way, along trajectories of the stochastic process, the function $q(t)$ is continuous and $p(t)$ presents jump discontinuities. We emphasize that even though we are facing a stochastic problem, the numerical simulation of these equations amounts to a sequence integrations of the standard, deterministic pendulum equation in (2.5), interspersed with changes (2.6) in the value of p at the impulse times.

It is clear that with the present setup, it is not possible for the random impulses to stabilize almost surely for $0 \le t < \infty$ the top-most, $q = 0$, equilibrium position of the inverted pendulum cf. [12]: if the waiting times τ_n are, say, exponentially distributed, there is a positive probability that the first impulse takes place once the pendulum has fallen. We shall prove below that it is however possible for the impulses to keep the pendulum up for arbitrarily long intervals $0 \le t \le T$ with probability arbitrarily close to 1.

3. Analysis

We begin by subtracting away the discontinuities in the angular velocity in (2.6) by means of the introduction of the new dependent variable

$$p_1 = p - \frac{v(t)}{\ell} \sin q. \tag{3.1}$$

It is perhaps of interest to observe that p_1 has a clear physical meaning (see e.g. [10, 11]): if m denotes the mass of the pendulum bob, the kinetic energy is

$$E_c = \frac{1}{2} m(\ell^2 p^2 - 2\ell v p \sin q + v^2)$$

and

$$\frac{\partial E_c}{\partial p} = m(\ell^2 p - \ell v \sin q),$$

so that, except for the factor $m\ell^2$, p_1 represents the generalized momentum conjugate to the generalized coordinate q. On the other hand, the change of variables $(p, q) \to (p_1, q)$ with p_1 given by (3.1) also arises as a first step in the application of the method of averaging to remove the explicit time-dependence of t in (2.1) (see e.g. Eq. (42) in [9], but there the acceleration $a(t)$ and the velocity $v(t)$ of S are smooth and deterministic).

In the variables (p_1, q), the equations of motion (2.5)–(2.6) become, for $t \neq t_n$,

$$\frac{dp_1}{dt} = -\frac{v(t)}{\ell} p_1 \cos q + \left(\frac{g}{\ell} - \frac{v^{*2}}{\ell^2} \cos q \right) \sin q, \qquad \frac{dq}{dt} = p_1 + \frac{v(t)}{\ell} \sin q; \tag{3.2}$$

now both dependent variables are continuous with jumps in the time-derivatives. In spite of the extra regularity that render it useful for the present analysis, the formulation (3.2) has the disadvantage when compared with (2.5)–(2.6) that the simple structure $dp/dt = \Phi(q)$, $dq/dt = p$ of (2.5) has been lost and, when performing numerical simulations, this precludes the use of a number of well-known integrators for oscillatory problems. This is not really important for the simple model problem considered here but would be in more realistic situations where numerical efficiency is an issue.

If the expected waiting time τ between shocks is seen as a small parameter, then $v(t)$ in (2.3) is an $O(1)$ quantity that changes sign rapidly and, in the spirit of the method of averaging, we may change variables to reduce the magnitude of the “oscillatory” terms that contain $v(t)$ in (3.2). The new variables (p_2, q_2) are formally defined by

$$p_1 = p_2 - \frac{s(t)}{\ell} p_2 \cos q_2, \qquad q = q_2 + \frac{s(t)}{\ell} \sin q_2, \tag{3.3}$$

where

$$s(t) = \int_0^t v(\zeta) d\zeta \tag{3.4}$$

is the displacement of the suspension point S. For τ is small, $\ell^{-1} s(t)$ will be small due to the oscillatory nature (2.3) of the function $v(t)$ being integrated (see the lemma below) and then the implicit function theorem guarantees that the change of variables (3.3) is well defined. (Note that, physically, the smallness of $\ell^{-1} s(t)$ means that the elongation of the suspension point is small with respect to the pendulum length.)

In the new variables (p_2, q_2), the equations of motion (3.2) become

$$\frac{dp_2}{dt} = \left(\frac{g}{\ell} - \frac{v^{*2}}{\ell^2}\cos q_2\right)\sin q_2 + R_p, \quad \frac{dq_2}{dt} = p_2 + R_q, \tag{3.5}$$

where the residuals R_p and R_q are smooth functions of p_2, q_2, $s(t)$, $v(t)s(t)$ that are formally $O(s(t))$. Thus the pendulum obeys a (time-dependent, stochastic) perturbed version of the smooth, deterministic, autonomous differential equations:

$$\frac{dP}{dt} = \left(\frac{g}{\ell} - \frac{v^{*2}}{\ell^2}\cos Q\right)\sin Q, \quad \frac{dQ}{dt} = P. \tag{3.6}$$

(We note in passing that the same equations are found in the study of the pendulum stabilization by vibration or noise. There v^{*2} represents the average of the square of the velocity $v(t)$ of the suspension point.)

Of course, the system (3.6) describes the motion of a particle in the so-called *effective* potential

$$V = \frac{g}{\ell}\cos Q - \frac{v^{*2}}{4\ell^2}\cos 2Q, \tag{3.7}$$

that possesses an absolute minimum at the lowest position of the pendulum $Q = \pm\pi$ and, for

$$v^{*2}/\ell > g, \tag{3.8}$$

also a local minimum at $Q = 0$. When this conditon holds, the topmost position of the pendulum provides a stable solution of the unperturbed system (3.6). In order to measure the deviation of the solutions of the system (3.5) — that includes the $O(s(t))$ perturbations R_p, R_q — from the solutions of (3.6), we have to obtain bounds for the displacement $s(t)$ of the suspension point S. In the deterministic case where the distribution F is concentrated at 1, so that $t_n = n\tau$, $s(t)$ is a saw-tooth function that attains its maximum value $v^*\tau$ at the odd-numbered impulse times t_{2n-1} and vanishes at the even-numbered t_{2n} (the downward trip of S between t_{2n-1} and t_{2n} exactly cancels the upward excursion between t_{2n-2} and t_{2n-1}). For genuinely random cases, with $\sigma^2 > 0$, there is only partial cancellation between the upward and downward excursions and the following lemma shows, that, as it may be expected, $s(t)$ behaves like $\tau^{1/2}$.

Lemma 3.1. *For arbitrary $T > 0$ and $\varepsilon > 0$,*

$$\mathbf{E}\left(\max_{0\le t\le T}|s(t)|^2\right) \le 4v^{*2}(\tau^2 + 2\sigma^2\tau\mathbf{E}(\tau N(T))).$$

and

$$\mathbf{P}\left\{\max_{0\le t\le T}|s(t)| \ge \varepsilon\right\} \le \varepsilon^{-2}v^{*2}(\tau^2 + 2\sigma^2\tau\mathbf{E}(\tau N(T))).$$

Proof. Set $s_n = s(t_n)$, $n = 1, 2, \ldots$, and note that, from (2.3) and (3.4), $s_n = v^* \hat{s}_n$, with

$$\hat{s}_n = \tau_1 - \tau_2 + \cdots + (-1)^{n-1} \tau_n,$$

and that, since $s(t)$ is piecewise linear,

$$\max_{0 \le t \le T} |s(t)| = \max\{|s_n| : 0 < t_n \le T\}.$$

The sequences $\{\hat{s}_{2n-1}\}$ and $\{\hat{s}_{2n}\}$ are martingales that are treated separately. For odd indices, standard (Doob) inequalities give

$$\mathbf{E}\left(\max_{1 \le k \le n} \hat{s}_{2k-1}^2\right) \le 4\mathbf{E}(\hat{s}_{2n-1}^2)$$

and

$$\mathbf{P}\left\{\max_{1 \le k \le n} \hat{s}_{2k-1}^2 \ge \varepsilon^2\right\} \le \varepsilon^{-2}\mathbf{E}(\hat{s}_{2n-1}^2),$$

where, on the right-hand sides,

$$\mathbf{E}(\hat{s}_{2n-1}^2) = (\mathbf{E}(\hat{s}_{2n-1}))^2 + \mathrm{Var}(\hat{s}_{2n-1}) = \tau^2 + (2n-1)\sigma^2\tau^2.$$

For even indices we proceed similarly, but in that case $\mathbf{E}(\hat{s}_{2n}^2) = 2n\sigma^2\tau^2$. We add the upper bounds of the odd and even cases to get

$$\mathbf{E}\left(\max_{1 \le k \le n} \hat{s}_k^2\right) \le 4v^{*2}(\tau^2 + 2n\sigma^2\tau^2)$$

and

$$\mathbf{P}\left\{\max_{1 \le k \le n} \hat{s}_k^2 \ge \varepsilon^2\right\} \le \varepsilon^{-2}v^{*2}(\tau^2 + 2n\sigma^2\tau^2).$$

The result now follows by conditioning to the value n of $N(T)$.

Even though it would be possible to provide a more detailed description of the behavior of the stochastic process $s(t)$, the bounds in the lemma are sufficient for our purposes and we turn to our main result:

Theorem 3.1. *Given a solution, $p(t)$, $q(t)$ of the stochastic equations* (2.5)–(2.6), *denote by $P(t)$, $Q(t)$ the solution of the deterministic, averaged system* (3.6) *with initial values $Q(0) = q(0)$, $P(0) = p(0) - \ell^{-1}v^* \sin q(0)$. Then there exist constants $C > 0$ and $\tau^{\max}$ such that, for $\tau \le \tau^{\max}$ and $\varepsilon > 0$*

$$\mathbf{E}\left(\max_{0 \le t \le T} |q(t) - Q(t)|^2\right) \le C(\tau^2 + \sigma^2\tau), \tag{3.9}$$

$$\mathbf{P}\left\{\max_{0 \le t \le T} |q(t) - Q(t)| < \varepsilon\right\} \ge 1 - C\varepsilon^{-2}(\tau^2 + \sigma^2\tau), \tag{3.10}$$

and, with p_1 *defined in* (3.1),

$$\mathbf{E}\left(\max_{0\le t\le T}|p_1(t)-P(t)|^2\right)\le C(\tau^2+\sigma^2\tau),$$

$$\mathbf{P}\left\{\max_{0\le t\le T}|p_1(t)-P(t)|<\varepsilon\right\}\ge 1-C\varepsilon^{-2}(\tau^2+\sigma^2\tau).$$

The constant C *depends only on* $p(0)$, $q(0)$, ℓ, g, v^* *and* T *(and is independent of the values of* ε, τ *and of the probability distribution* F*) and* $\tau^{\max}$ *depends on* $p(0)$, $q(0)$, ℓ, g, T, v^* *and* F *(and is independent of* ε, τ*).*

Proof. In view of (2.4), for small τ (how small depends on T and on the specific distribution F), the quantity $\mathbf{E}(\tau N(T))$ in the lemma is less than, say, $2T$. On the other hand, by the implicit function theorem applied to (3.3), the differences $|p_1(t)-p_2(t)|$ and $|q(t)-q_2(t)|$ can be bounded, uniformly in $0\le t\le T$, by a constant factor of $\max_{0\le t\le t}|s(t)|$ and, by standard perturbation results applied to (3.6), the same can be said for the differences $|q_2(t)-Q(t)|$ and $|p_2(t)-P(t)|$.

When the stability condition (3.8) holds, solutions of (3.6) with $Q(0)$ and $P(0)$ close to 0 remain for all time in a well of the effective potential V. Then the theorem ensures that, for fixed v^* and in a given bounded interval $0\le t\le T$, solutions of (2.5)–(2.6) with small initial conditions will remain small with high probability, provided that τ is suitably small.

4. Some Simulations

In all the numerical simulations to be reported, we set $\ell = 0.2$ m, $g = 9.8$ ms^{-2} and $v^* = 3\sqrt{\ell g}$, so that the stability requirement (3.8) is satisfied. For this choice of parameters, the period $2\pi\sqrt{\ell/g}$ of the small oscillations of the (un-shocked) pendulum around the down-most equilibrium at $q=\pi$ is $\approx 1s$; note also that v^* is of the order of magnitude of the maximum velocity $2\sqrt{\ell g}$ attained by the (un-shocked) pendulum when falling freely after being abandoned near its top-most position $q = 0$. The initial conditions are taken throughout to be $p(0) = 0$ and $q(0) = 0.6$.

The simulation in Fig. 1 corresponds to the interval $0\le t\le T=2$ with waiting times τ_n drawn from the exponential distribution with expectation $\tau = 0.01$. The upper part of the figure depicts $q(t)$ and the discontinuous $p(t)$ along the sample path: the shocks have stabilized the pendulum at the top equilibrium, in spite of the fact that the initial deviation from the upward vertical was rather large. The lower left corner shows the continuous momentum variable defined in (3.1). When this is represented against the angle q (bottom right), we obtain a coarse approximation to the phase portrait of (3.6) near the stable equilibrium at the origin.

Our analysis shows that, if v^* is kept constant as it is the case in the simulations presented here, our system converges in the limit $\tau\to 0$ to the deterministic system (3.6). Figure 2 is similar to Fig. 1, but now the impulses are very frequent, $\tau = 0.01/128$, and the graphs of q and p_1 as functions of t possess a much smoother

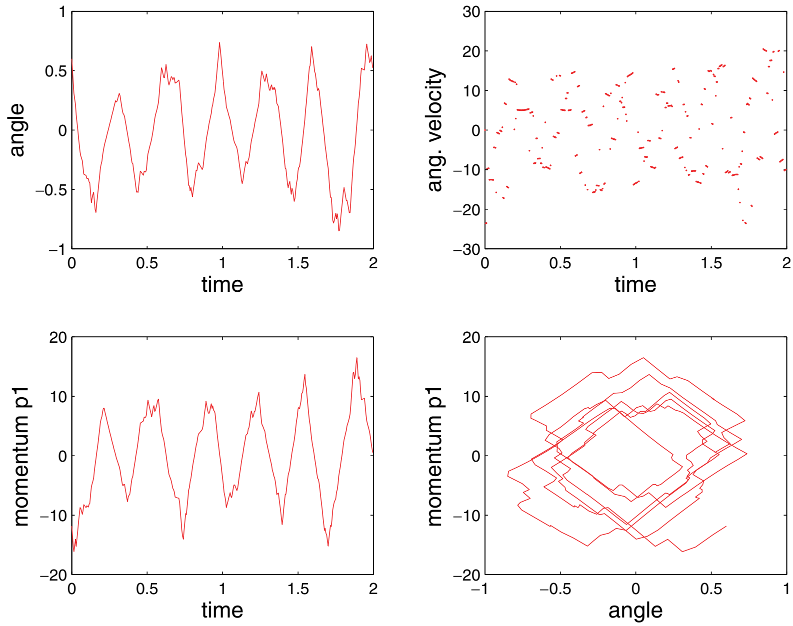

Fig. 1. A sample path with $\tau = 0.01$.

appearance. For this small value of τ numerical experiments show that, in agreement with our analysis, the pendulum typically remains near $q = 0$ for longer time intervals than it does for coarser choices of τ. However, to produce Fig. 2, we have experimented with many choices for the initial seed of the random number generator, so as to obtain a trajectory that shows a remarkable behavior. The simulation takes place up to $T = 4$. The pendulum falls down shortly after $t = 1$, makes a complete turn around the suspension point S and then starts swinging around the down-most equilibrium at $q = -\pi$. The phase plane (bottom right) shows clearly that the dynamics of the pendulum with shocks is a perturbed version of the dynamics of a particle in the (effective) potential (3.7) with wells at 0, $\pm\pi$.

Given in Fig. 3 is the expectation on the left-hand side of (3.9) when $T = 0.1$ as a function of $\tau = 0.01, 0.01/2, 0.01/4, \ldots$. The expectation is computed by averaging over 1000 sample paths. There are four choices for the distribution function F: (1) exponential (variance $\sigma^2 = 1$), (2) uniform in the interval $[0, 2]$ ($\sigma^2 = 1/3$), (3) the discrete distribution with $\mathbf{P}\{1/2\} = \mathbf{P}\{3/2\} = 1/2$ ($\sigma^2 = 1/4$), (4) deterministic ($\sigma^2 = 0$). The $O(\tau)$ behavior guaranteed by (3.9) is clearly borne out in the three random cases, while the deterministic simulation yields an $O(\tau^2)$ behavior, in agreement with the discussion that precedes the lemma (alternatively set $\sigma^2 = 0$ in (3.9)). Note additionally that the spacing between the three lines arising from the random distributions correspond to the ratios $1 : 1/3 : 1/4$ between the variances, as predicted by (3.9).

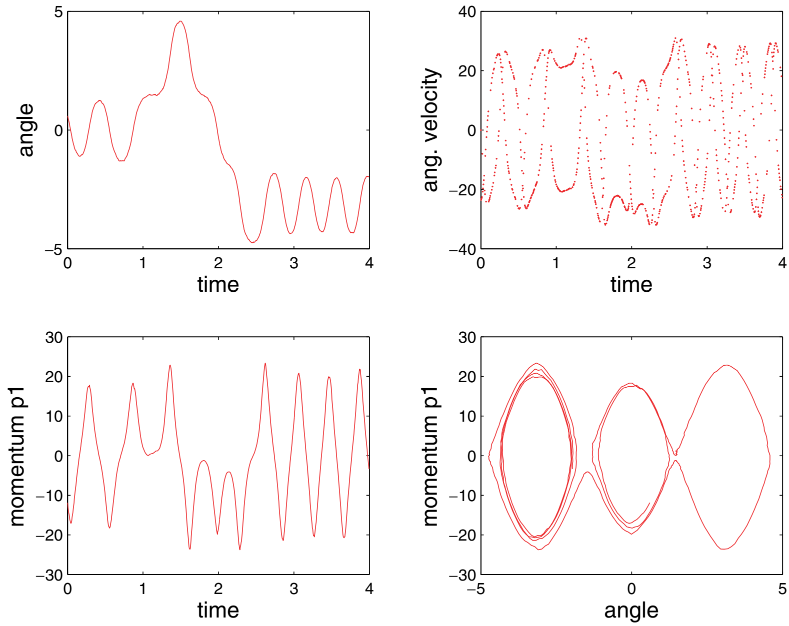

Fig. 2. A sample path with $\tau = 0.01/128$.

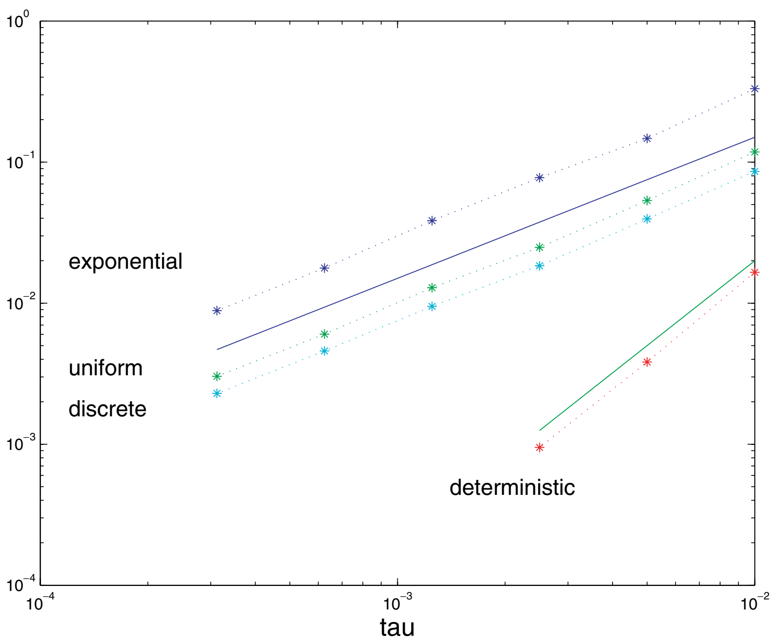

Fig. 3. The expectation of $\max_t |q(t) - Q(t)|^2$ as a function of the expected waiting time τ.

5. Concluding Remarks

The above material may be extended in many directions. For instance, standard Lyapunov techniques, may be applied to improve the result of the Theorem when the initial condition is in a well of the effective potential.

In addition to the stabilization by fast vibrations ($\tau \ll 1$) of small amplitude ($\ell^{-1}s(t) \ll 1$) considered here and easily analyzed by averaging, one may expect that there may be additional stable regimes [7, 11]. This issue has not been investigated.

Finally, it is clear that the format (2.2) for the impulsive acceleration of the suspension point is not the only possible and other choices should be explored. Alternative choices, including more parameters, would be particularly helpful in connection with the study of situations where the impulses happen more and more frequently and one may expect to obtain a limit system governed by a stochastic equation. With the choice (2.2) used here, the variance of $s(t)$ behaves essentially as $v^{*2}\tau t$; thus, unless v^* is (un-physically) allowed to grow without bound, any limit with $\tau \to 0$ will be deterministic. Furthermore, with the choice (2.2), the process $s(t)$ is not stationary and therefore does not fit in the framework considered in, say, [12].

Acknowledgments

I am indebted to discussions with A. Debussche, D. Higham, X. Mao, C. Matrán and A. M. Stuart. My research is supported by project MTM 2007–63257, DGI, MEC, Spain.

References

1. L. Arnold, H. Crauel and V. Wihstutz, Stabilization of linear systems by noise, *SIAM J. Control Optim.* **21** (1983) 451–461.
2. L. Arnold, Stabilization by noise revisited, *Z. Argew. Math. Mech.* **70** (1990) 235–246.
3. F. Colonius and W. Kliemann, *The Dynamics of Control* (Birkhäuser, 2000).
4. H. Crauel, T. Damm and A. Ilchmann, Stabilization of linear systems by rotation, *J. Diff. Eqns.* **234** (2007) 412–438.
5. E. Faou and T. Lelièvre, Conservative stochastic differential equations: Mathematical and numerical analysis, preprint.
6. L. Grüne, Asymptotic controllability and exponential stabilization of nonlinear control systems at singular points, *SIAM J. Control Optim.* **36** (1998) 1585–1603.
7. P. Imkeller and Ch. Lederer, Some formulas for Lyapunov exponents and rotation numbers in two dimensions and the stability of the harmonic oscillator and the inverted pendulum, *Dyn. Syst.* **16** (2001) 29–61.
8. P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer, 1992).
9. M. Levi, Geometry and physics of averaging with applications, *Physica D* **132** (1999) 150–164.
10. G. J. Mata and E. Pestana, Effective Hamiltonian and dynamic stability of the inverted pendulum, *Eur. J. Phys.* **25** (2004) 717–721.

11. R. Mitchell, Stability of the inverted pendulum subjected to almost periodic and stochastic base motion — an application of the method of averaging, *Int. J. Nonlinear Mech.* **7** (1972) 101–123.
12. A. I. Ovseyevich, The stability of an inverted pendulum when there are rapid random oscillations of the suspension point, *J. Appl. Math. Mech.* **70** (2006) 762–768.
13. J. M. Sanz-Serna and A. M. Stuart, Ergodicity of dissipative differential equations subject to random impulses, *J. Diff. Eqns.* **155** (1999) 262–284.