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CARRYING AN INVERTED PENDULUM ON A BUMPY ROAD

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Dedicated to P. Kloeden on his 60th birthday

Abstract. We study the stabilization by means of random impulses of an unstable linear oscillator. Almost sure exponential stability is proved for some combinations of the parameter values.

1. Introduction.

We study the stabilization by means of random impulses of the unstable equilibrium $q = 0$ of the equation

$$\frac{d^2q}{dt^2} = -\nu \frac{dq}{dt} + \frac{\ell}{g} q, \tag{1}$$

that describes the motion of a linearized mathematical pendulum with friction. Here $\nu > 0$ is a friction coefficient, $\ell > 0$ the pendulum length, $g > 0$ the acceleration of gravity and q the angle between the pendulum rod and the *upward* vertical axis.

Unstable equilibria of linear or nonlinear, damped or undamped deterministic oscillators may become stable when the system is subjected to a vibrating external forcing term (vibrational stabilization in the terminology of [1]). This fact was first discovered experimentally by Stephenson in 1908 for the case of an inverted pendulum whose pivot is subjected to fast *vertical* vibrations and has led to several useful physical applications, including the Nobel-prize winning Paul's trap (a brief historical summary may be seen in [3]).

Stochastic forcing terms have been known since the 1960s [6] to possess similar stabilizing properties (stabilization by noise); many useful references are provided in [2].

As in [8], we consider here *impulsive* external forces of random amplitude that act on (1) after random waiting times [9]. While the results in [8] apply only on bounded time-intervals, our analysis, based on [7], shows almost sure exponential stability on $0 < t < \infty$ [4].

In order to provide a pictorial description of the problem to be treated, we may think of an inverted pendulum that is transported on a carriage (Figure 1). From time to time a bump in the road is encountered and this causes a shock that instantaneously increments the vertical velocity of the carriage and feeds energy to its suspension mechanism. The shocks and subsequent oscillations of the carriage

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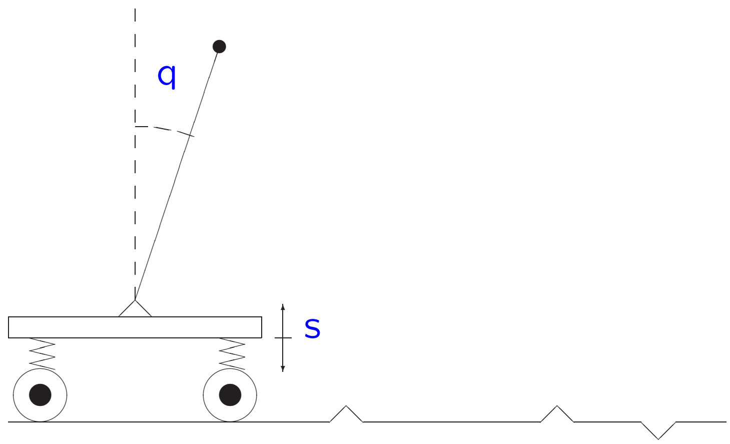

FIGURE 1. The inverted pendulum on a carriage that travels on a bumpy road.

are in turn transferred to the inverted pendulum and we wish to study the long-time behavior of the resulting random dynamical system.[1]

Our main result is given in Section 3. Sections 2 and 4 are respectively devoted to present preliminary material and numerical simulations.

2. The motion of the carriage.

2.1. **Equations of motion.** The carriage is subjected to vertical shocks that occur at times $t_0 = 0 < t_1 < \cdots < t_n < \dots$ that are assumed to follow a Poisson process with intensity $1/\tau$. In the intervals $t_n < t < t_{n+1}$ between consecutive impulses the (upwards) vertical displacement $s(t)$ of the carriage from its equilibrium position and the corresponding velocity $v(t)$ obey the differential equations of a damped harmonic oscillator,

$$\frac{dv}{dt} = -\gamma v - ks, \qquad \frac{ds}{dt} = v, \qquad t_n < t < t_{n+1}, \quad n = 0, 1, \dots; \tag{2}$$

the damping coefficient γ and the spring constant k are both strictly positive. At the impulse times the displacement remains continuous

$$s(t_n) = s(t_n+) = s(t_n-), \qquad n = 0, 1, \dots, \tag{3}$$

but the velocity is not defined and possesses jump discontinuities

$$v(t_n+) = v(t_n-) + w\theta_n, \qquad n = 0, 1, \dots, \tag{4}$$

where θ_n is a dimensionless random variable and w a parameter with dimensions of velocity that governs the strength of the shocks. The variables θ_n in (4) are supposed to be mutually independent and independent of the Poisson process and share a common distribution with

$$E(\theta_n) = 0, \qquad E(\theta_n^2) = 1, \qquad E(\theta_n^3) = 0, \qquad E(\theta_n^4) < \infty. \tag{5}$$

[1]However note that a realistic description of the inverted pendulum would require to use the nonlinear version of (1) with $\sin q$ instead of q. It is not possible [8] to obtain almost sure stability for the nonlinear situation. In the nonlinear case, once the pendulum is far away from the equilibrium at $q = 0$, friction is likely to drive it to the equilibrium at the bottom position $q = \pm\pi$.

The equations (2)–(4) have to be supplemented with the specification of the vector $(v(0-), s(0))$ of (random) initial values, which is assumed to be independent of the t_n and of the θ_n.

2.2. **The associated Markov chain.** We begin by considering (cf. [9]) the variables $v_n = v(t_n-)$, $s_n = s(t_n)$, $n = 0, 1, \ldots$, that form a discrete-time, homogeneous Markov chain whose state space is the plane $\mathcal{R}^2$. Our first two results provide the equations for the evolution of the distributions of this chain and the corresponding characteristic functions.

Lemma 2.1. *Assume that the distributions of (v_n, s_n) have densities $\phi_n(v, s)$. Then*

$$\phi_{n+1}(v,s) + \tau\frac{\partial}{\partial v}[(-\gamma v - ks)\phi_{n+1}(v,s)] + \tau\frac{\partial}{\partial s}[v\phi_{n+1}(v,s)] = \int_{-\infty}^{\infty} \phi_n(v - v', s)\, \mu\left(\frac{dv'}{w}\right),$$

where μ is the probability law of the variables θ_n in (4).

Proof. Denote by $\psi_n(v, s, t)$ the probability density for the deterministic equation (2) corresponding to the initial density

$$\psi_n(v,s,0) = \int_{-\infty}^{\infty} \phi_n(v - v', s)\, \mu\left(\frac{dv'}{w}\right)$$

of the random vector $(v(t_n+), s(t_n))$. Then

$$\phi_{n+1}(v,s) = \tau^{-1}\int_0^{\infty} \exp(-t/\tau)\psi_n(v,s,t)\, dt \tag{6}$$

so that $\tau\phi_{n+1}$ is the Laplace transform (with dual variable $1/\tau$) of ψ_n. The result is now obtained by taking Laplace transforms in the Liouville equation satisfied by ψ_n. ☐

Lemma 2.2. *The characteristic function $\Phi_{n+1}(\xi, \eta)$ of (v_n, s_n) can be obtained from Φ_n by means of the partial differential equation*

$$\Phi_{n+1} + \tau\gamma\xi\frac{\partial \Phi_{n+1}}{\partial \xi} + \tau k\xi\frac{\partial \Phi_{n+1}}{\partial \eta} - \tau\eta\frac{\partial \Phi_{n+1}}{\partial \xi} = g(w\xi)\Phi_n, \tag{7}$$

where g is the characteristic function of the distribution of the random variables θ_n in (4).

Proof. It is sufficient to consider the case where the distributions of (v_n, s_n) possess densities. In that case, (7) is a direct consequence of Lemma 1. ☐

Setting $\Phi_n = \Phi_{n+1} = \Phi$ in (7), yields the (singular) first-order partial differential equation

$$\Phi + \tau\gamma\xi\frac{\partial \Phi}{\partial \xi} + \tau k\xi\frac{\partial \Phi}{\partial \eta} - \tau\eta\frac{\partial \Phi}{\partial \xi} = g(w\xi)\Phi, \tag{8}$$

whose characteristic system

$$\frac{d\xi}{d\sigma} = \gamma\xi - \eta, \quad \frac{d\eta}{d\sigma} = k\xi.$$

has $\xi = 0$, $\eta = 0$ as an unstable equilibrium. Each point (ξ_0, η_0) of the plane of the dual Fourier variables (ξ, η) can be joined to the origin $\xi = 0$, $\eta = 0$ by means of a unique characteristic curve $(\xi(\sigma), \eta(\sigma))$ with $(\xi(-\infty), \eta(-\infty)) = (0, 0)$,

$(\xi(0),\eta(0)) = (\xi_0,\eta_0)$ and by integrating along such a characteristic, it is easily seen that solutions of (8) satisfy

$$\Phi(\xi_0,\eta_0) = \exp\left(\tau^{-1}\int_{-\infty}^{0}[-1+g(w\xi(\sigma))]\,d\sigma\right)\Phi(0,0).$$

For a characteristic function, $\Phi(0,0) = 1$ and we conclude:

Lemma 2.3. *The Markov chain (v_n, s_n) has a unique invariant distribution. The corresponding characteristic function is the unique solution of the partial differential equation (8) that satisfies the condition $\Phi(0,0) = 1$.*

The proof of our next result is a simple calculation using differentiation in (8) and taking into account that, from (5), $g'(0) = 0$, $g''(0) = -1$.

Lemma 2.4. *The invariant distribution of the chain (v_n, s_n) has finite moments of orders ≤ 4 and*

$$E(v_n) = E(s_n) = 0,$$

$$E(v_n^2) = \frac{w^2}{2\gamma\tau},\quad E(v_n s_n) = 0,\quad E(s_n^2) = k^{-1}E(v_n^2).$$

2.3. **The carriage stationary process.** We leave the discrete-time chain (v_n, s_n) and take up the study of the continuous-time process $(v(t-), s(t))$ (whose paths are left-continuous with right limits) or, equivalently, its càdlàg modification $(v(t+), s(t))$ (where paths are right-continuous with left limits).

Lemma 2.5. *For fixed $t \geq 0$ and conditional on t_n being the last impulse time $< t$, the vector $(v(t-), s(t))$ has the same distribution as the vector (v_{n+1}, s_{n+1}).*

Proof. For fixed $t \geq 0$, consider the last impulse time $t_n \leq t$. Due to well-known properties of the exponential distribution, the spent waiting time $t - t_n$ is exponentially distributed with expectation τ and hence, under the hypothesis of Lemma 2.1, the right hand-side of (6) represents the density of $(v(t-), s(t))$. Therefore this density coincides with that of (v_{n+1}, s_{n+1}). ☐

From now on we assume that the initial data $(v(0-), s(0))$ for the equations (2)–(4) of the carriage motion are drawn from the invariant probability distribution of the Markov chain. Lemma 2.5 and the lack of memory of the exponential waiting times imply that, as t varies, $(v(t-), s(t))$ is a continuous-time, càglàd stationary process and that, for each fixed value of t, the distribution of the vector $(v(t-), s(t))$ is that given in Lemma 2.3.

The existence and uniqueness of a stationary solution of the equations of motion of the carriage, proved here by explicit computations, may alternatively be established by noting that (v, s) obey a two-dimensional Ornstein-Uhlenbeck process driven by a Lévy process (more precisely by the compound Poisson process $w\sum_{t_n\leq t}\theta_n$). The general theory of such OU processes also shows, [5] Theorem 4.3, that the stationary solution is *ergodic* (cf. [9]) and even β-mixing.

2.4. **Some auxiliary processes.** We now introduce two processes that will be used in the next section. For $t \geq 0$, we denote by $J(t)$ the smallest jumping time t_n such that $t_n > t$ and set

$$S(t) = \frac{1}{2\ell^2\gamma}(v(t-)^2 + ks(t)^2) - \frac{E(v^2)}{\ell^2}(J(t) - t), \qquad (9)$$

and

$$M(t) = -\sum_{t_n \le t} \frac{\Delta v(t_n)^2}{2\ell^2\gamma} + \frac{E(v^2)}{\ell^2} J(t). \tag{10}$$

Furthermore for $t \ge 0$, we consider the σ-algebra $\mathcal{F}_t$ generated by the variables $(v(t'-), s(t'))$ and $J(t')$, $0 \le t' \le t$ (note that for this filtration, the waiting time $t_{n+1} - t_n$ is 'known' at t_n). The following result holds.

Lemma 2.6. *(i) The process $S(t)$ is stationary.*

(ii) In the intervals $t_n < t < t_{n+1}$,

$$\frac{dS}{dt} = -\ell^{-2}(v^2 - E(v^2)). \tag{11}$$

(iii) At the jumping times $t_n > 0$,

$$\begin{aligned} -\Delta S(t_n) = \Delta M(t_n) &= -\frac{\Delta v(t_n)^2}{2\ell^2\gamma} + \frac{E(v^2)}{\ell^2}(t_{n+1} - t_n) \\ &= \frac{1}{2\ell^2\gamma}\left[-2w\theta_n v(t_n-) - w^2\theta_n^2 + w^2\frac{t_{n+1} - t_n}{\tau}\right]. \end{aligned} \tag{12}$$

(iiii) $M(t)$ is a càdlàg process whose paths are constants in the intervals $[t_n, t_{n+1})$. Furthermore $M(t)$ is a square-integrable martingale with respect to $\mathcal{F}_t$.

Proof. For (i), note that $J(t) - t$ is stationary due to the lack of memory of the exponential waiting times. Parts (ii) and (iii) are simple computations using the definitions of S and M along with (2) and Lemma 2.4. (Note that at $t_0 = 0$, the left limits of S and M and hence ΔS and ΔM are not defined.)

Conditioned to $\mathcal{F}_t$, $t < t_n$, the expectation of $2w\theta_n v(t_n-) + w^2\theta_n^2$ is w^2 (see (5)) and the expectation of $t_{n+1} - t_n$ equals τ. Therefore the conditional expectation of the right-most term in (12) vanishes and M is a martingale. □

2.5. **Scaling.** When an inverted pendulum is subjected to a *deterministic* harmonic vibration, so that the vertical velocity of the pivot is of the form $v(t) = v_{max}\cos(2\pi t/\tau)$, stability is only achieved if the period τ is sufficiently small. In the stochastic analysis to be presented in the next section, it is therefore necessary to maintain the expected waiting time τ between consecutive impulses as a *free (small) parameter*. As τ decreases, the kinetic energy per unit time pumped into the carriage by the shocks increases and in order to keep a suitable balance between noise and dissipation it is necessary to vary appropriately the coefficients γ and k in (2). Based on dimensionality considerations, we set

$$\gamma = \gamma^*\tau^{-1}, \qquad k = k^*\tau^{-2},$$

where γ^* and k^* are arbitrary but fixed positive constants. In this way the stationary process for the carriage is of the form

$$v(t) = v^*(\tau^{-1}t), \qquad s(t) = \tau s^*(\tau^{-1}t),$$

where (v^*, s^*) is the (τ-independent) stationary process for the oscillator

$$\frac{dv^*}{dt^*} = -\gamma^* v^* - k^* s^*, \qquad \frac{ds^*}{dt^*} = v^*,$$

when the impulses arrive with intensity 1. In what follows, we keep the physically meaningful variables v, s, t, etc. rather than using their mathematically-scaled counterparts v^*, s^*, t^*, etc.

3. The motion of the pendulum.

3.1. Equations of motion.

3.1. Equations of motion. The pendulum differential equation reads (cf. (1)):

$$\frac{d^2q}{dt^2} = -\nu \frac{dq}{dt} + \ell^{-1}(g + a(t))q, \quad t_n < t < t_{n+1}, \quad n = 0, 1, \ldots, \tag{13}$$

where $a(t) = dv/dt$ is the acceleration of the carriage.[2] We introduce the angular velocity $p = dq/dt$ as a new dependent variable and rewrite (13) as a first order system

$$\frac{dp}{dt} = -\nu p + \ell^{-1}(g + \frac{dv}{dt})q, \qquad \frac{dq}{dt} = p, \qquad t_n < t < t_{n+1}, \quad n = 0, 1, \ldots \tag{14}$$

At the impulse times the angular velocity, which is not defined, follows the jumps in v and we have

$$p(t_n+) = p(t_n-) + \ell^{-1}\Delta v(t_n) q(t_n-), \qquad n = 0, 1, \ldots, \tag{15}$$

with $\Delta v(t_n) = v(t_n+) - v(t_n-)$, while the angle remains continuous

$$q(t_n+) = q(t_n-), \qquad n = 0, 1, \ldots \tag{16}$$

3.2. Main result. For the long-time behavior of the inverted pendulum we have the following result.

Theorem 3.1. *Consider the process defined by (14)–(16), where v is the stationary process described above and $p(0-)$, $q(0)$ are deterministic initial values and assume that $E(v^2)/\ell > g$. Then there exists a constant τ_0 (depending on w, γ^*, k^*, ℓ, g and the distribution of θ_1) such that for $\tau < \tau_0$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log |q(t)| < 0, \quad \text{a.s.}$$

Remark 1. According to Lemma 2.4, the condition $E(v^2)/\ell > g$ is equivalent to $w^2/(\gamma\tau) = w^2/\gamma^* > 2\ell g$, so that the ratio of the strength w of the impulses to the square root of the (non-dimensional) friction coefficient γ^* of the carriage has to be larger than $\sqrt{2\ell g}$. ($\sqrt{2\ell g}$ has the dimensions of a velocity. When a nonlinear mathematical pendulum of length ℓ is abandoned from the top-most position $q = 0$, it reaches the bottom position $q = \pi$ with velocity $\sqrt{2\ell g}$.) For the physics of the condition $E(v^2)/\ell > g$ see [3].

Proof of the theorem. The notation $O(\tau^m)$ will be used to refer to a process whose absolute value possesses a bound of the form $\tau^m \xi(t)$, where $\xi \geq 0$ is a stationary integrable process independent of τ. Thus the pivot acceleration $a(t)$, velocity $v(t)$ and displacement $s(t)$ are respectively $O(\tau^{-1})$, $O(1)$, $O(\tau)$. The process $S(t)$ introduced in (9) is $O(\tau)$ and so are the jumps of the martingale M in (10).

As in [7] or [8], the behavior of the paths $(p(t), q(t))$ is investigated by performing a sequence of (invertible, symplectic) changes of variables in the spirit of the method of averaging.

We first use the change

$$p = p_1 + \ell^{-1} v q_1, \qquad q = q_1,$$

[2] We work under the reasonable hypothesis that the mass of the pendulum is so small that the pendulum does not influence the motion of the carriage.

(with $O(1)$ coefficients) so as to remove the jump discontinuities present in p (see (15)). This transforms (14) into the system

$$\frac{dp_1}{dt} = -\nu p_1 + \ell^{-1}(g - \ell^{-1}v^2)q_1 - \ell^{-1}vp_1 - \nu\ell^{-1}vq_1,$$
$$\frac{dq_1}{dt} = p_1 + \ell^{-1}vq_1,$$

where we note the centrifugal acceleration $\ell^{-1}v^2$ that opposes g (see the discussion in [3]). Next the 'oscillatory' terms $\nu\ell^{-1}vq_1$, $\ell^{-1}vp_1$, $\ell^{-1}vq_1$ (with $O(1)$ coefficients) are removed by means of the successive substitutions[3]

$$p_1 = p_2 - \nu\ell^{-1}sq_2, \qquad q_1 = q_2,$$

and

$$p_2 = \frac{1}{\chi(\ell^{-1}s)}p_3, \qquad q_2 = \chi(\ell^{-1}s)q_3, \qquad \chi(\sigma) = 1 + \sigma\mathrm{sech}(\sigma),$$

that lead to a system of the form

$$\frac{dp_3}{dt} = -\nu p_3 + \ell^{-1}(g - \ell^{-1}v^2)q_3 + O(\tau)p_3 + O(\tau)q_3, \tag{17}$$
$$\frac{dq_3}{dt} = p_3 + O(\tau)p_3 + O(\tau)q_3.$$

The functions $p_i(t)$, $q_i(t)$, $i = 1, 2, 3$, are continuous and their derivatives have jumps at the impulse times t_n; the corresponding differential equations only hold in the open intervals $t_n < t < t_{n+1}$.

Finally, we take

$$p_3 = p_4 + S(t)q_4, \qquad q_3 = q_4$$

where S is the stationary process in (9). In view of (11), this change of variables essentially has the effect of replacing the process v^2 in (17) by its expectation $E(v^2)$:

$$\frac{dp_4}{dt} = -\nu p_4 - \Lambda q_4 + O(\tau)p_4 + O(\tau)q_4, \quad \Lambda = -\ell^{-1}(g - \ell^{-1}E(v^2)) > 0,$$
$$\frac{dq_4}{dt} = p_4 + O(\tau)p_4 + O(\tau)q_4.$$

At each t_n, p_4 is discontinuous in view of the continuity of p_3 and of the jumps in S (see (12)) and furthermore, for $t_n > 0$,

$$\Delta p_4(t_n) = -\Delta S(t_n)q_4(t_n) = \Delta M(t_n)q_4(t_n). \tag{18}$$

The proof is concluded by showing that $|q_4(t)|$ decays exponentially to 0 with probability 1. We simplify the notation by omitting henceforth the subscripts in p_4 and q_4. The quadratic form (Lyapunov function)

$$V = \frac{1}{2}p^2 + \frac{\nu}{2}pq + \left(\frac{\nu^2}{4} + \frac{\Lambda}{2}\right)q^2$$

is positive definite and satisfies

$$\frac{dV}{dt} = -\frac{\nu}{2}(p^2 + \Lambda q^2) + BO(\tau), \qquad t_n < t < t_{n+1},$$

[3]The 'simpler' change $p_2 = \exp(-\ell^{-1}s)p_3$, $q_2 = \exp(\ell^{-1}s)q_3$ used in [7] has the drawback of requiring, in view of the growth of the exponential function, the introduction of additional hypotheses on the process s. The exact form of χ here is not important provided that χ, χ' and $1/\chi$ are uniformly bounded and $\chi(\sigma) = 1 + \sigma + O(\sigma^2)$ near $\sigma = 0$.

where B is a quadratic form in the variables p, q. Therefore, there exist positive constants κ and C_1 and a stationary process $\xi \geq 0$ with finite expectation such that

$$\frac{d}{dt}\log V = \frac{1}{V}\frac{dV}{dt} \leq -\kappa + C_1\tau\xi(t), \qquad t_n < t < t_{n+1},$$

and integration, taking into account the jumps, leads to

$$\log V(t+) - \log V(0+) \leq -\kappa t + C_1\tau\int_0^t \xi(\sigma)\,d\sigma + \sum_{0<t_n\leq t}\Delta\log V(t_n).$$

We estimate the jumps $\Delta\log V(t_n)$ as follows:

$$\begin{aligned}\Delta\log V(t_n) &\leq \frac{\Delta V(t_n)}{V(t_n-)}\\ &= \frac{1}{V(t_n-)}\left[\frac{\partial V}{\partial p}\Big|_{t_n-}\Delta p(t_n) + \frac{1}{2}(\Delta p(t_n))^2\right]\\ &= \frac{q\frac{\partial V}{\partial p}}{V}\Big|_{t_n-}\Delta M(t_n) + \frac{1}{2}\frac{q(t_n)^2}{V(t_n-)}[\Delta M(t_n)]^2.\end{aligned}$$

Here we have successively used the concavity of the logarithm, the Taylor expansion of the quadratic form V, and (18). Thus

$$\begin{aligned}t^{-1}\left(\log V(t+) - \log V(0+)\right) \leq\; & -\kappa + C_1\tau\frac{1}{t}\int_0^t \xi(\sigma)\,d\sigma\\ & +\frac{1}{t}\sum_{0<t_n\leq t}\frac{q\frac{\partial V}{\partial p}}{V}\Big|_{t_n-}\Delta M(t_n)\\ & +C_2\tau\frac{1}{t}\sum_{0<t_n\leq t}\frac{[\Delta M(t_n)]^2}{\tau},\end{aligned} \tag{19}$$

where C_2 is a positive constant and we are left with the task of estimating the last three terms in the right hand-side. By the ergodic theorem, with probability 1,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \xi(\sigma)\,d\sigma \to E(\xi),$$

and therefore, for t large and τ small, the term with the integral in (19) is $< \kappa/3$. A similar argument applies to the last term. The first sum in (19) is an Ito integral of a bounded integrand with respect to the martingale $M(t) - M(0)$, and therefore, as t increases, grows more slowly than $(\tau t)^{(1/2)+\epsilon}$, for any $\epsilon > 0$. This concludes the proof. □

4. **Some simulations.** In this section we present some numerical simulations. In all of them, the pendulum parameters are $\nu = 5$, $g = 9.8$, $\ell = 0.20$, while the carriage suspension has $\gamma^* = 2$, $k^* = 2$. This leaves the waiting time τ between impulses and the strength of the bumps w as free parameters. The variable θ_1 takes the values ± 1 with probability 1/2 each. The initial condition is $p(0-) = 0$, $q(0) = 0.1$.[4]

For each choice of the values of τ and w, we computed numerically 50 samples of the process for $0 \leq t \leq 40$. A sample was taken to be 'stable' if $|q(40)|/|q(0)| < 0.0001$. The stars in Figure 2 correspond to those combinations of τ and w for which all 50 samples were 'stable'. For fixed w, our Theorem (see Remark 1)

[4]Since the model is linear the value of $q(0)$ plays no decisive role.

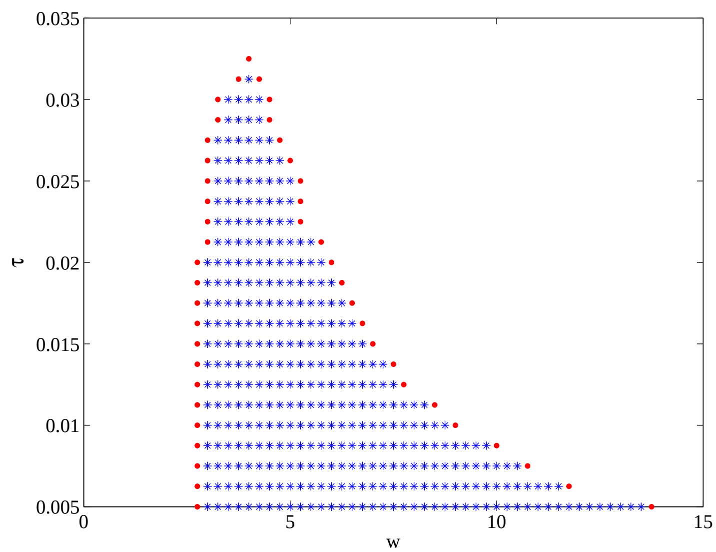

FIGURE 2. Combinations of τ and w that ensure stabilization. Experimental results.

ensures stabilization, for τ sufficiently small, provided that $w > (2\ell g\gamma^*)^{1/2} = 2.8$; such a stabilization is clearly manifest in the figure. On the other hand, if we look at a fixed value of τ, we see that the stabilization is achieved only on a bounded range $(w_{\min}, w_{\max})$: if w is too small the energy pumped into the system is not enough to achieve the stabilization, if w is too large, the occurrence of a bump that moves the pendulum away from $q = 0$ becomes more likely. As τ decreases, the interval $(w_{\min}, w_{\max})$ tends to $((2\ell g\gamma^*)^{1/2}, \infty)$.

In the bound (19), the bumps contribute to the jumps $\Delta M(t_n)$ that feature in both summations. The second summation, where all terms being added are positive, offers a larger threat to stability than the first (in this connection recall from the analysis that the first grows with t more slowly than the second). Therefore it may be conjectured that the destabilizing effect of bumps is related to the size of the second summation in (19). To check this conjecture we have tracked the values of the relevant quantities and found, after tedious computations,

$$\begin{aligned} \kappa &= \nu\left(1 - \frac{\nu}{\sqrt{4\Lambda + \nu^2}}\right), \\ C_2 &= \frac{4}{4\Lambda + \nu^2}, \\ E([\Delta M(t_n)]^2) &= \frac{2+\gamma^*}{4\ell^4\gamma^{*3}}\tau^2 w^4. \end{aligned}$$

We then looked for the combinations of τ and w for which in the right hand-side of (19)

$$C_2\tau\frac{1}{t}\left(\frac{t}{\tau}\frac{E([\Delta M(t_n)]^2)}{\tau}\right) < \frac{\kappa}{3}.$$

The boundary of the resulting stability region is depicted in Figure 3; the qualitative resemblance with Figure 2 is apparent. The values of τ in Figure 3, that caters for a

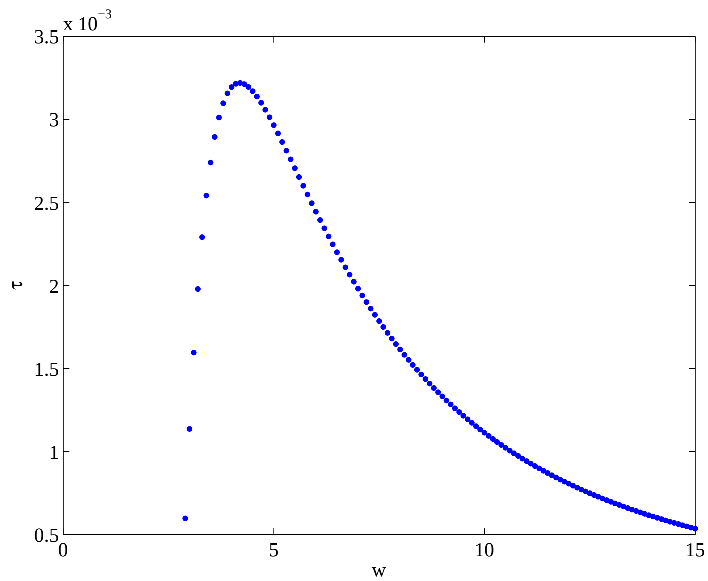

FIGURE 3. Combinations of τ and w that ensure stabilization. Estimated values using the proof of the theorem.

worst case scenario, are one order of magnitude smaller than those of the simulations in Figure 2.

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